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TESIS DOCTORAL

Systems of Markov type functions. Normality and convergence of Hermite-Padé approximants.

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UNIVERSIDAD CARLOS III DE MADRID

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Resumen

En 1873, Charles Hermite publicó en [44], utilizando técnicas de aproximación racional simultánea a sistemas de funciones exponenciales, la demostración de la trascendencia de e. Años después, en 1892, Carl Louis Ferdinand von Lindeman extendió el trabajo de Hermite para probar la trascendencia de π. Los aproximantes Hermite-Padé reciben este nombre en honor a Hermite y al matemático francés Henri Padé quien en su tesis doctoral [62], bajo la dirección de Hermite estudió con detalle esta clase de funciones racionales.

El problema que analizó Padé puede enunciarse de la siguiente forma.
Dada una función f, analítica en un entorno del punto $z = 0$ y dados un par de números naturales n y m encontrar unos polinomios $P_{n,m}$ y $Q_{n,m}$ tales que:

i) *deg* $P_{n,m} \leq n$, *deg* $Q_{n,m} \leq m \quad Q_{n,m} \not\equiv 0$,

ii) $Q_{n,m}(z)f(z) - P_{n,m}(z) = \mathcal{O}(z^{n+m+1}) \qquad z \to 0.$

Esta construcción define una única función racional $R_{n,m}(f) = \frac{P_{n,m}}{Q_{n,m}}$ llamada aproximante de Padé de tipo (n, m) de la función f.

Posteriormente se realizaron diversas extensiones de este problema. En una de ellas se trata de aproximar varias funciones simultáneamente, dando lugar a los aproximantes Hermite-Padé de tipo I y de tipo II.

Definición (Aproximantes Hermite-Padé de tipo I). *Sea* $\mathbf{f} = (f_1, \ldots, f_m)$ *un vector de funciones analíticas en un dominio* Ω *del plano complejo que contiene al infinito y* $\mathbf{n} = (n_1, \ldots, n_m) \in \mathbb{Z}_+^m \setminus \{\mathbf{0}\}$ *un multi-índice fijado, aquí* $\{\mathbf{0}\}$ *denota el vector nulo en* $\mathbb{Z}_+^m$. *Sea* $|\mathbf{n}| = n_1 + \ldots + n_m$. *A los polinomios* $(a_{\mathbf{n},0}, a_{\mathbf{n},1}, \ldots, a_{\mathbf{n},m})$ *que cumplen*

i) *deg* $a_{\mathbf{n},j} \leq n_j - 1, \quad j = 1, \ldots, m$ *donde* $(a_{\mathbf{n},j} \equiv 0$ *cuando deg* $a_{\mathbf{n},j} = -1)$.

ii) $\sum_{j=1}^{m} a_{\mathbf{n},j}(z)f_j(z) - a_{\mathbf{n},0}(z) = \mathcal{O}(1/z^{|n|}), \qquad z \to \infty.$

para un cierto polinomio $a_{\mathbf{n},0}$, los llamaremos aproximantes Hermite-Padé de tipo I de $\mathbf{f}$ *respecto al multi-índice* $\mathbf{n}$.

Definición (Aproximantes Hermite-Padé de tipo II)**.** *Sea* $\mathbf{f} = (f_1, \ldots, f_m)$ *un vector de funciones analíticas en un dominio* Ω *del plano complejo que contiene al infinito y* $\mathbf{n} = (n_1, \ldots, n_m) \in \mathbb{Z}_+^m \setminus \{\mathbf{0}\}$ *un multi-índice fijado entonces existen polinomios* $Q_{\mathbf{n}}$ *y* $(P_{\mathbf{n},1}, \ldots P_{\mathbf{n},m})$ *que cumplen*

i) $deg\ Q_{\mathbf{n}} \leq |n| = n_1 + \ldots + n_m, \quad Q_{\mathbf{n}} \not\equiv 0$

ii) $Q_{\mathbf{n}}(z) f_j(z) - P_{\mathbf{n},j}(z) = \mathcal{O}(1/z^{n_j+1}), \qquad z \to \infty, \qquad j = 1, \ldots, m$

Al vector de funciones racionales $(\frac{P_{\mathbf{n},1}}{Q_{\mathbf{n}}}, \ldots \frac{P_{\mathbf{n},m}}{Q_{\mathbf{n}}})$*, lo llamaremos aproximante Hermite-Padé de tipo II de* $\mathbf{f}$ *respecto al multi-índice* $\mathbf{n}$.

Observese que ambas definiciones coinciden, en el caso de trabajar con una sola función $f(z)$. Además, podemos fusionar estas dos definiciones en la siguiente.

Consideremos un multi-índice de la forma

$$\mathbf{n} = (\mathbf{n_1}, \mathbf{n_2}) \in \mathbb{Z}_+^{m_1} \times \mathbb{Z}_+^{m_2}, \quad \text{con } |\mathbf{n_1}| = |\mathbf{n_2}| + 1,$$

donde $\mathbf{n_1} = (n_{1,1} \ldots, n_{1,m_1})$ y $\mathbf{n_2} = (n_{2,1} \ldots, n_{2,m_2})$.

Definición (Aproximantes Hermite-Padé de tipo mixto)**.** *Sea* $\mathbf{F}$ *una matriz de dimensión* $m_2 \times m_1$ *de funciones analíticas en un dominio* Ω *del plano complejo que contiene al infinito y* $\mathbf{n}$ *un multi-índice dado. Al vector de polinomios* $\mathbf{A_n} = (a_{\mathbf{n},1}, \ldots, a_{\mathbf{n},m_1})$ *que cumple*

i) $\mathbf{A_n} \not\equiv 0$*, deg* $a_{\mathbf{n},j} \leq n_{1,j} - 1,$ *para todo* $j = 1, 2, \ldots, m_1$.

ii) $\left(\mathbf{FA}_{\mathbf{n}}^T - \mathbf{D}_{\mathbf{n}}^T\right)(z) = \left(\mathcal{O}(1/z^{n_{2,1}+1}), \ldots, \mathcal{O}(1/z^{n_{2,m_2}+1})\right)^T, \qquad z \to \infty$

para un cierto vector de polinomios $\mathbf{D_n} = (d_{\mathbf{n},1}, d_{\mathbf{n},2}, \ldots, d_{\mathbf{n},m_2})$*, lo llamaremos aproximante Hermite-Padé de tipo mixto de* $\mathbf{F}(z)$ *respecto al multi-índice* $\mathbf{n}$.

La existencia del vector de polinomios $\mathbf{A_n} = (a_{\mathbf{n},1}, \ldots, a_{\mathbf{n},m_1})$ queda garantizada pues es equivalente a encontrar una solución no trivial a un sistema de ecuaciones lineales homogéneo con $|\mathbf{n_1}|$ incógnitas (los coeficientes de $\mathbf{A_n}$) y $|\mathbf{n_2}|$ ecuaciones (dadas por las condiciones *ii)*), con lo cual siempre existe una solución no trivial.

Los casos particulares $m_1 = 1$ y $m_2 = 1$ son los aproximantes de Padé de tipo II y de tipo I respectivamente.

Además de los resultados de Hermite, los aproximantes de Padé de tipo I, tipo II y de tipo mixto han sido usados en la demostración de la irracionalidad de otros números. Por ejemplo, F.Beukers en [12] muestra que la prueba de Apery de la irracionalidad de $\zeta(3)$, (ver [5]), puede encuadrarse en el contexto de la aproximación Hermite-Padé mixta.

Diferentes aplicaciones de los aproximantes Hermite-Padé en la física, la química, el análisis numérico, la estadística y la economía, entre otros, pueden encontrarse en [9, 10].

Definición. *Un multi-índice* $\mathbf{n}$ *se dice normal para la matriz de funciones* $\mathbf{F}(z)$ *para la aproximación Hermite-Padé de tipo mixto si* $\quad deg\ a_{\mathbf{n},j} = n_{1,j} - 1 \quad$ *para todo* $j = 1, 2, \ldots, m_1$. *La matriz de funciones* $\mathbf{F}(z)$ *se dice perfecta si todos los multi-índices*

$$\mathbf{n} = (\mathbf{n_1}, \mathbf{n_2}) \in \mathbb{Z}_+^{m_1} \times \mathbb{Z}_+^{m_2}, \quad \textit{tales que } |\mathbf{n_1}| = |\mathbf{n_2}| + 1,$$

son normales.

La unicidad de los aproximantes Hermite-Padé y la normalidad de los multi-índices para determinadas clases de funciones juegan un papel fundamental, tanto desde el punto de vista teórico como de las aplicaciones. En el caso de una función, ya comentamos que el aproximante de Padé es único, sin embargo en los caso de tipo I, de tipo II y de tipo mixto no se tiene que cumplir esta propiedad.

En el Capítulo 2 estudiamos los aproximantes Hermite-Padé de tipo mixto para una clase amplia de matrices de funciones analíticas. Entre los resultados expuestos, responderemos a la cuestión de la unicidad de los aproximantes Hermite-Padé de tipo mixto y la normalidad de los multi-índices para la clase de funciones analizadas, además obtendremos varias propiedades adicionales que nos servirían en el desarrollo de futuras lineas de investigación relacionadas con esta temática.

En [4], A. Angelesco analizó la aproximación Hermite-Padé de tipo II a un sistema de funciones específico, posteriormente conocido como sistema Angelesco, y demostró que este sistema de funciones es perfecto (ver también [58]). Dicho sistema utiliza funciones de Markov de la forma $f_j(z) = \int_{\Delta_j} \frac{d\mu_j(x)}{z-x}$ para $j = 1, 2, \ldots, m$ donde los Δ_j son intervalos disjuntos dos a dos y μ_j es una medida positiva soportada en Δ_j.

A pesar de que la normalidad tipo II para los sistemas de Angelesco es fácil de deducir, los polinomios multiortogonales asociados no tienen un buen comportamiento asintótico, (ver [6, 40] y [42]). En este sentido, resulta mucho más interesante, desde un punto de vista geométrico y analítico, otros tipos de sistemas de m funciones introducido por E.M. Nikishin en [59].

Sean Δ_α y Δ_β dos intervalos acotados de forma que $\Delta_\alpha \cap \Delta_\beta = \emptyset$. Dadas dos medidas σ_α y σ_β soportadas respectivamente en Δ_α y Δ_β, podemos definir una tercera medida $\langle \sigma_\alpha, \sigma_\beta \rangle$ cuya forma diferencial es

$$d\langle \sigma_\alpha, \sigma_\beta \rangle(x) = \int \frac{d\sigma_\beta(t)}{x-t} d\sigma_\alpha(x) = \widehat{\sigma}_\beta(x) d\sigma_\alpha(x).$$

Definición. *Sea* $\Delta_j, j = 1, \ldots, m$, *una colección de intervalos acotados tales que*

$$\Delta_j \cap \Delta_{j+1} = \emptyset, \qquad j = 1, \ldots, m-1.$$

Sea $(\sigma_1, \ldots, \sigma_m)$ *un sistema de medidas tales que* $Co(\mathrm{supp}(\sigma_j)) = \Delta_j, \sigma_j \in \mathcal{M}(\Delta_j)$, $j = 1, \ldots, m$. *Decimos que* $\mathrm{s} = (s_{1,1}, \ldots, s_{1,m}) = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$, *donde*

$$s_{1,1} = \sigma_1, \quad s_{1,2} = \langle \sigma_1, \sigma_2 \rangle, \ldots \quad , s_{1,m} = \langle \sigma_1, \langle \sigma_2, \ldots, \sigma_m \rangle \rangle$$

es un sistema de medidas de Nikishin generado por $(\sigma_1, \ldots, \sigma_m)$.

En [59] se demuestra que todos los multi-indices en $\mathbb{Z}_+^m$ de la forma

$$(n+1, \ldots, n+1, n, \ldots, n)$$

son normales tipo II y se señala sin demostración que son normales todos los multi-indices tales que $n_1 \geq \ldots \geq n_m$.

En [26], K. Driver y H. Stahl prueban que son normales de tipo II todos los multi-indices tales que $1 \leq j < k \leq m$ implica que $n_k \leq n_j + 1$. Posteriormente los mismos autores demostraron la normalidad tipo I para los mismos multi-indices (ver también [27, 28]).

En [32] (ver también [33]), U. Fidalgo y G. López Lagomasino probaron que los sistemas de Nikishin son perfectos para la aproximación Hermite-Padé de tipo mixto que incluye como casos extremos los tipo I y tipo II.

Los primeros estudios sobre la convergencia de los aproximantes de Padé, se llevaron a cabo a finales del siglo XIX por los matemáticos P. L. Tchebycheff (véase por ejemplo [77, 78, 79], A. A. Markov [57] y T. J. Stieltjes [75, 76]. En particular los teoremas de Markov y Stieltjes en esta dirección juegan un papel motivador en esta memoria.

Sea s una medida tal que $x^n \in L_1(s)$ para todo $n \in \mathbb{Z}_+$. Por Δ denotemos la envoltura convexa del soporte de la medida s y por $\hat{s}$ la transformada de Cauchy de la medida s. dada por

$$\hat{s}(z) = \int \frac{ds(x)}{z-x}.$$

Obviamente $\hat{s} \in \mathcal{H}(\bar{\mathbb{C}} \setminus \Delta)$, la clase de funciones holomorfas en $\bar{\mathbb{C}} \setminus \Delta$. Cuando Δ es acotado se dice que $\hat{s}$ es una función tipo Markov, y tipo Stieltjes si Δ es no acotado.

Sea $\{c_n\}_{n\geq 0}$ una sucesión de números reales. Se dice que el problema de momentos para la sucesión $\{c_n\}_{n\geq 0}$ está definido en $\mathbb{R}_+$ si existe una medida s cuyo soporte está contenido en $\mathbb{R}_+$ tal que

$$c_n = \int x^n ds(x), \quad n = 0, 1, \ldots$$

a c_n se le llama momento enésimo de s. Si s es única se dice que el problema de momentos está determinado. Una condición suficiente para que un problema de momentos (definido en $\mathbb{R}_+$) esté determinado es que se cumpla la condición de Carleman (ver [17]) dada por

$$\sum_{n\geq 0} |c_n|^{-1/2n} = \infty.$$

El teorema de Stieltjes dice lo siguiente. Sea s una medida cuyo soporte está contenido en $\mathbb{R}_+$ y $\{c_n\}_{n\geq 0}$ la sucesión de su momentos. Sea $\{P_n/Q_n\}_{n\geq 0}$ la sucesión de los aproximantes de Padé de $\hat{s}$ $(n = m)$. Entonces

$$\lim_{n\to\infty} \frac{P_n}{Q_n}(z) = \widehat{s}(z), \tag{1}$$

uniformemente en cada conjunto compacto K contenido en $\mathbb{C} \setminus \Delta$ si y solo si el problema de momentos para $\{c_n\}_{n\geq 0}$ está determinado.

Si Δ está acotado, combinando el teorema de Weirstrass sobre densidad de los polinomios en $C(\Delta)$ y el teorema de dualidad de Riesz, se deduce que en este caso (1) tiene lugar. Este resultado se conoce como teorema de Markov.

Para los aproximantes Hermite-Padé de tipo II de un sistema de medidas de Niksihin $(s_{1,1}, \ldots, s_{1,m}) = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$ resultados análogos al teorema de Markov y de Stieltjes han sido obtenidos (ver, por ejemplo [15, 30, 31, 42, 72]. En terminos generales en estos trabajos se demuestra bajo hipótesis adecuadas que las sucesiones diagonales de aproximantes Hermite-Padé del sistema de funciones $(\hat{s}_{1,1}, \ldots, \hat{s}_{1,m})$ convergen componente a componente a dicha función vectorial uniformemente sobre cada subconjunto compacto de $\bar{\mathbb{C}} \setminus \Delta_1$. El objetivo fundamental del Capítulo 3 es obtener un resultado análogo para aproximantes Hermite-Padé de tipo I. En contraste con lo que sucede con los tipo II probamos que las sucesiones diagonales de aproximantes Hermite-Padé tipo I del sistema $(\hat{s}_{1,1}, \ldots, \hat{s}_{1,m})$ convergen uniformemente en cada subconjunto compacto de $\bar{\mathbb{C}} \setminus \Delta_m$ al vector de funciones $(\hat{s}_{m,m}, \ldots, \hat{s}_{m,1})$, donde $(s_{m,m}, \ldots, s_{m,1}) = \mathcal{N}(\sigma_m, \ldots, \sigma_1)$. Este fenomeno no había sido observado anteriormente y, posiblemente, constituye el resultado más singular de esta memoria.

En un intento por extender el teorema de Markov a una clase general de funciones meromorfas, A.A. Gonchar considero funciones de la forma $\widehat{s} + r$ donde r es un función racional cuyos polos están en $\mathbb{C} \setminus \Delta$. En [36], Gonchar demostró que si Δ es un intervalo acotado y $s' > 0$ casi dondequiera en Δ, entonces se tiene convergencia uniforme de los aproximantes de Padé de estas funciones meromorfas y probó adicionalmente que cada polo de r en $\mathbb{C} \setminus \Delta$ "atrae" tantos ceros de Q_n como su orden y que el resto de los ceros de Q_n se acumulan en Δ cuando $n \to \infty$. Después, en [65] E.A. Rakhmanov obtuvo una extensión completa del teorema de Markov cuando r tiene coeficientes reales demostrando que si r tiene coeficientes complejos entonces dicho resultado no es posible sin imponer condiciones adicionales sobre la medida s.

En el Capítulo 4 consideramos un vector de funciones racionales $\mathbf{r} = (r_1, \ldots, r_m) = \left(\frac{v_1}{t_1}, \ldots, \frac{v_m}{t_m}\right)$, de forma que $\deg t_j = d_j$ y $\deg v_j < d_j$, para todo $j = 1, \ldots, m$. Ahora tomamos un sistemas de funciones meromorfas de la forma $\mathbf{f} = (f_1, \ldots, f_m) = \widehat{\mathbf{s}} + \mathbf{r}$, donde

$$f_j(z) = \widehat{s}_{1,j}(z) + r_j(z), \qquad j = 1, \ldots, m.$$

y $\mathbf{s} = (s_{1,1}, \ldots, s_{1,m})$ es un sistema de Nikshin.

En [15, 31] se dan teoremas de tipo Stieltjes para los aproximantes de Hermite-Padé de tipo II de sistemas de Nikishin. Para el caso de sistemas de Nikishin generados por medidas de soporte compacto en [30, 42] (también [31]) se precisa la velocidad de convergencia. En el Capítulo 4 estudiaremos la convergencia de los aproximantes Hermite-Padé tipo II para sistemas de Nikishin perturbados mediantes fracciones racionales de la forma $\mathbf{f} = \widehat{\mathbf{s}} + \mathbf{r}$.

Para este tipo de pertubación racional de sistemas de Nikishin el único estudio hasta la fecha se encuentra en [16]. Ese trabajo se restringe a sistemas de Nikishin de 2 medidas y fracciones racionales con coeficientes reales. Nosotros consideramos sistemas de Nikishin generales de m medidas. En el caso que $\mathbf{r}$ tiene coeficientes reales y $m = 2$ nuestro resultado principal del Capítulo 4 recupera el probado en [16]. Para $\mathbf{r}$ con coeficientes complejos, suponiendo que $\Delta_j, j = 1, \ldots, m$ es un intervalo acotado y $\left|\sigma_j'\right| > 0$ casi dondequiera en Δ_j también probamos convergencia de los aproximantes de Hermite-Padé de tipo II de $\mathbf{f} = \hat{\mathbf{s}} + \mathbf{r}$ pero en un sentido más debil que el uniforme sobre subconjuntos compactos.

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Chapter 1

Introduction

«CONTENTS»

- Brief introduction to Hermite-Padé approximants.
- Preliminaries.
- Structure of the thesis.

THIS thesis deals with Hermite-Padé approximation of analytic and merophorphic functions. As such it is embeded in the theory of vector rational approximation of analytic functions which in turn is intimately connectd with the theory of multiple orthogonal polynomials. All the basic concepts and results used in this thesis involving complex analysis and measure theory may found in classical textbooks such as [1, 70].

In 1873, Charles Hermite published in [44] his proof of the transcendence of e making use of simultaneous rational approximation of systems of exponentials. That paper marked the beginning of the modern analytic theory of numbers. Since their introduction by Ch. Hermite, these approximants have been employed in the proof of the irrationality and trascendency of other numbers. For example, in [12] F. Beukers shows that Apery's proof (see [5]) of the irrationality of $\zeta(3)$ can be placed in the context of mixed type Hermite-Padé approximation. See [80] for a brief introduction and survey on the subject. More recently, mixed type approximation has appeared in random matrix and non-intersecting brownian motions theories (see, for example, [14, 21, 47, 48]).

C.Hermite
(1822-1901)

In this chapter we give a brief introduction to Hermite-Padé approximants, and define some important concepts. In Section 1.3 we explain the main results that were obtained in this work.

1.1 Some historical remarks

The formal theory of simultaneous rational approximation for general systems of analytic functions was initiated by K. Mahler in lectures delivered at the University of Groningen in 1934-35. These lectures were published years later in [56]. Important contributions in this respect are also due to his students J. Coates and H. Jager, see [18] and [45]. K. Mahler's approach to the simultaneous approximation of finite systems of analytic functions may be reformulated in the following terms.

Let $\mathbf{f} = (f_1, \ldots, f_m)$ be a family of analytic functions in some domain D of the extended complex plane containing ∞. Fix a non-zero multi-index $\mathbf{n} = (n_1, \ldots, n_m) \in \mathbb{Z}_+^m \setminus \{\mathbf{0}\}, |\mathbf{n}| = n_1 + \ldots + n_m$. Here, $\{\mathbf{0}\}$ denotes the null element in $\mathbb{Z}_+^m$ and $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$. There exist polynomials $(a_{\mathbf{n},1}, \ldots, a_{\mathbf{n},m})$, not all identically equal to zero, such that

i) $\deg a_{\mathbf{n},j} \leq n_j - 1, j = 1, \ldots, m$ ($\deg a_{\mathbf{n},j} = -1$ means that $a_{\mathbf{n},j} \equiv 0$),

ii) $\sum_{j=1}^m a_{\mathbf{n},j}(z) f_j(z) - a_{\mathbf{n},0}(z) = \mathcal{O}(1/z^{|\mathbf{n}|}), z \to \infty,$

for some polynomial $a_{\mathbf{n},0}$. Analogously, there exists $Q_\mathbf{n}$, not identically equal to zero, such that

i') $\deg Q_\mathbf{n} \leq |\mathbf{n}|,$

ii') $Q_\mathbf{n}(z) f_j(z) - P_{\mathbf{n},j}(z) = \mathcal{O}(1/z^{n_j+1}), z \to \infty, \quad j = 1, \ldots, m,$

for some polynomials $P_{\mathbf{n},j}, j = 1, \ldots, m.$

The existence of the vector of polynomials $(a_{\mathbf{n},1}, \ldots, a_{\mathbf{n},m})$ reduces to solving a homogeneus linear system of $|\mathbf{n}| - 1$ equations on the total number of $|\mathbf{n}|$ coefficients of $(a_{\mathbf{n},1}, \ldots, a_{\mathbf{n},m})$, and the existence of $Q_\mathbf{n}$ reduces to solving a homogeneus linear system of $|\mathbf{n}|$ equations on the total number of $|\mathbf{n}| + 1$ coefficients of the polynomial $Q_\mathbf{n}$; therefore, a non-trivial solution is guaranteed. The polynomials $a_{\mathbf{n},0}$ and $P_{\mathbf{n},j}, j = 1, \ldots, m,$ are uniquely determined from *ii)* and *ii')* once their partners are found.

Initially, the polynomials $(a_{\mathbf{n},1},\ldots,a_{\mathbf{n},m})$ were called latin and $Q_{\mathbf{n}}$ german polynomials, due to the letters employed in denoting them (see the papers of Mahler, Coates and Jager cited above).

Later, the two constructions have been called type I and type II polynomials (approximants) of the system $(f_1,\ldots,f_m)$. Algebraically, the two constructions are closely related. This is clearly exposed in [18, 45, 56, 61]. When $m=1$ both definitions coincide with that of the well-known Padé approximation in its linear presentation.

K. Mahler introduced the concept of perfect systems in the general theory that he developed for the simultaneous Hermite-Padé approximation of analytic functions.

In applications in the areas of number theory, convergence of simultaneous rational approximation, and asymptotic properties of type I and type II polynomials, a central question is if these polynomials have no defect; that is, if they attain the maximal degree possible.

Definition. *A multi-index* $\mathbf{n}$ *is said to be* **normal** *for the system* $\mathbf{f}$ *for type I approximation if i)-ii) implies that* $\deg a_{\mathbf{n},j} = n_j - 1, j = 1,\ldots,m$*, and is said to be* **normal** *for the system* $\mathbf{f}$ *for type II approximation if i')-ii') implies that* $\deg Q_{\mathbf{n}} = |\mathbf{n}|$*. A system of functions* $\mathbf{f}$ *is said to be* **perfect** *if all multi-indices are normal.*

It is easy to verify that $(a_{\mathbf{n},1},\ldots,a_{\mathbf{n},m})$ and $Q_{\mathbf{n}}$ are uniquely determined to within a constant factor when $\mathbf{n}$ is type I or type II normal respectively. The convenience of these properties is quite clear. For example, a normalization allows to determine these entities in a unique form.

Considering the construction at the origin (instead of $z=\infty$ which we chose for convenience), the system of exponentials considered by Hermite, $(e^{w_1 z},\ldots,e^{w_m z})$, $w_i \neq w_j, i \neq j, i,j = 1,\ldots,m$, is known to be perfect for type I and type II. A second example of a perfect system for both types is that given by the binomial functions $((1-z)^{w_1},\ldots,(1-z)^{w_m}), w_i - w_j \notin \mathbb{Z}, i \neq j$. All multi-indices $\mathbf{n}$ such that $n_1 \geq \cdots \geq n_m$ are known to be type I and type II normal for $(\log^m(1-z),\ldots,\log(1-z),1)$. Systems satisfying this property are called weakly perfect. Basically, these are the only examples known of perfect and weakly perfect systems, except for certain ones formed by Cauchy transforms of measures.

It is well known that Padé and Hermite-Padé approximation have a close relation with the theory of orthogonal polynomials (as is shown in Sections 1.2 and 2.4 below).

One of the main goals of this thesis is to reveal large classes of type I and type II perfect systems of functions. This is carried out in Chapter 2 in a general setting called mixed type Hermite-Padé approximation which contains type I and type II as special cases.

One of the basic general results on the convergence of diagonal Padé approximants is the clasical theorem of Markov (see [57]). Markov´s theorem states that

$$\lim_{n\to\infty} \frac{P_n(z)}{Q_n(z)} = \widehat{s}(z)$$

uniformly on each compact subset of $\overline{\mathbb{C}}\setminus\Delta$, where $\widehat{s}$ is the Cauchy transform of a measure s whose support is contained on a real bounded interval Δ of the real line $\mathbb{R}$ (see Section 1.2) and P_n/Q_n is the $n-$thiagonal Padé approximant with respect to $\widehat{s}$. When Δ is an unbounded interval a similar theorem was obteined by Stieljtes (see [75]) in terms of an associated moment problem.

A.A.Markov
(1856-1922)

Let $\{c_n\}_{n\geq 0}$ be a sequence of real numbers. We say that the moment problem for the sequence $\{c_n\}_{n\geq 0}$ is defined on $\mathbb{R}_+$ if there exists a measure s whose support is contained in $\mathbb{R}_+$ such that

$$c_n = \int x^n ds(x), \quad n = 0, 1, \ldots.$$

c_n is called n-th moment of s. If s is unique we say that the moment problem is determinate. A sufficient condition for the moment problem to be determinate (on $\mathbb{R}_+$) is that

$$\sum_{n\geq 0} |c_n|^{-1/2n} = \infty.$$

This is known as Carleman's condition (see [17]). Stieltjes' theorem states the following. Let s be a measure whose support is contained in $\mathbb{R}_+$ and $\{c_n\}_{n\geq 0}$ the sequence of its moments. Let $\{P_n/Q_n\}_{n\geq 0}$ be the sequence of diagonal Padé approximants of $\hat{s}$ $(n = m)$. Then

$$\lim_{n\to\infty} \frac{P_n}{Q_n}(z) = \widehat{s}(z), \tag{1.1}$$

uniformly on each compact subset K contained on $\mathbb{C}\setminus\Delta$ if and only if the moment problem for the sequence $\{c_n\}_{n\geq 0}$ is determinate.

If Δ is bounded, combining Weirstrass' theorem on the density of polynomials in $C(\Delta)$ and Riesz's duality theorem it follows that the moment problem is determinate and (1.1) follows. Hence, Markov´s theorem is a consequence of Stieltjes´ result.

Other extensions in the context of classical and multipoint diagonal Padé approximation can be found in [36, 37, 39, 50, 51, 52, 53, 54, 65]. In the case of Hermite-Padé approximation analogous results appear in [15, 16, 30, 31, 42, 59].

In Chapter 3 and Chapter 4 we obtain different extensions for type I and type II Hermite-Padé approximation of certain classes of meromorphic functions obtained by rational modifications of Nikishin systems.

1.2 Markov systems and orthogonality

Let s be a finite Borel measure with constant sign whose support consists of infinitely many points and is contained in the real line. When the support $\operatorname{supp}(s)$ of s is unbounded we assume additionally, that all the moments of s are finite; that is, $c_n = \int x^n ds(x) < \infty$, $n \in \mathbb{Z}_+$. In the sequel, we only consider such measures. By Δ we denote the smallest containing interval $\operatorname{supp}(s)$. We denote this class of measures by $\mathcal{M}(\Delta)$. Let

$$\widehat{s}(z) = \int \frac{ds(x)}{z-x}$$

denote the Cauchy transform of s. Obviously, $\widehat{s} \in \mathcal{H}(\overline{\mathbb{C}} \setminus \Delta)$; that is, it is analytic in $\overline{\mathbb{C}} \setminus \Delta$. Moreover

$$\widehat{s}(z) \sim \sum_{j=0}^{\infty} \frac{c_j}{z^{j+1}}, \qquad c_j = \int x^j ds(x). \tag{1.2}$$

If Δ is bounded the series is convergent in a neighborhood of ∞; otherwise, the expansion is an asymptotic one at ∞. That is, for each $k \geq 0$

$$\lim_{z\to\infty} z^{k+1} \left(\widehat{s}(z) - \sum_{j=0}^{k-1} \frac{c_j}{z^{j+1}} \right) = c_k,$$

where the limit is taken along any curve which is non tangential to $\operatorname{supp}(s)$ at ∞.

If we apply the construction above to the system formed by $\widehat{s}$ $(m=1)$, it is easy to verify that $Q_{\mathbf{n}}$ turns out to be orthogonal to all polynomials of degree less than n

$\in \mathbb{Z}_+$. Consequently, $\deg Q_{\mathbf{n}} = n$, all its zeros are simple and lie in the open convex hull $\mathrm{Co}(\mathrm{supp}(s))$ of $\mathrm{supp}(s)$. These properties allow to deduce Markov's and Stieltjes' theorems on the convergence of diagonal Padé approximations of $\widehat{s}$. For this reason, $\widehat{s}$ is often called a Markov function when $\mathrm{supp}(s)$ is bounded and a Stieltjes function when the support is unbounded.

Cauchy transforms of measures are quite relevant in several respects. Many elementary functions can be expressed as such. The resolvent function of a self-adjoint operator admits this type of representation. If one allows complex weights, any reasonable analytic function in the extended complex plane with a finite number of algebraic singularities adopts that form. This fact, and the use of Padé approximation, has played a central role in some of the most important achievements in the last decades concerning the exact rate of convergence of best rational approximation: namely, A.A. Gonchar and E.A. Rakhmanov's result, see [38, 41] and [7], on the best rational approximation of e^{-x} on $[0, +\infty)$; and H. Stahl's theorem, see [73], on the best rational approximation of x^α on $[0, 1]$.

Let us see two other examples of general systems of Markov functions much more illustrative for our purpose.

1.2.1 Angelesco systems.

In [4], A. Angelesco considered the following systems of measures.

Definition 1.2.1. *A system of measures* $\mathbf{S} = (s_1, \ldots, s_m)$ *is an Angelesco system when* $s_j \in \mathcal{M}(\Delta_j)$, $j = 1, \ldots, m$, *and the* Δ_j *are pairwise disjoint bounded intervals.*

Fix $\mathbf{n} \in \mathbb{Z}_+^m \setminus \{\mathbf{0}\}$ and consider the type II approximant of the so called Angelesco system of functions $(\widehat{s}_1, \ldots, \widehat{s}_m)$ relative to $\mathbf{n}$. It turns out that

$$\int x^\nu Q_{\mathbf{n}}(x) ds_j(x) = 0, \quad \nu = 0, \ldots, n_j - 1, \quad j = 1, \ldots, m.$$

Therefore, $Q_{\mathbf{n}}$ has n_j simple zeros in the interior (with respect to the euclidean topology of $\mathbb{R}$) of Δ_j. In consequence, since the intervals Δ_j are pairwise disjoint, $\deg Q_{\mathbf{n}} = |\mathbf{n}|$.

Therefore $(\hat{s}_1, \ldots, \hat{s}_m)$ is type II perfect. We also say that the Angelesco system $(s_1, \ldots, s_m)$ itself is type II perfect.

Unfortunately, Angelesco's paper received little attention and such systems reappear many years later in [58], where E.M. Nikishin deduces some of their formal properties.

Though type II normality for Angelesco systems is easy to deduce, the multiple orthogonal polynomials and the rational approximations associated with them do not have good asymptotic behavior. In [6] and [40], their logarithmic and strong asymptotic behavior, respectively, are given. From the results in [40], it follows that type II Hermite-Padé approximation of Angelesco systems, in general, do not converge uniformly on compact subsets of the complement of the support of the measures. In this respect, a different system of Markov functions turns out to be much more interesting and foundational from the geometric and analytic points of view.

1.2.2 Nikishin systems.

In an attempt to construct general classes of functions for which normality takes place, in [59] E.M. Nikishin introduced the concept of MT-systems (now called Nikishin systems).

Let us introduce what is called a Nikishin system of measures. Let $\Delta_\alpha, \Delta_\beta$ be two intervals contained in the real line which have at most one point in common, $\sigma_\alpha \in \mathcal{M}(\Delta_\alpha), \sigma_\beta \in \mathcal{M}(\Delta_\beta)$, and $\widehat{\sigma}_\beta \in L_1(\sigma_\alpha)$. With these two measures we define a third one as follows (using the differential notation)

$$d\langle \sigma_\alpha, \sigma_\beta \rangle(x) := \widehat{\sigma}_\beta(x) d\sigma_\alpha(x).$$

Above, $\widehat{\sigma}_\beta$ denotes the Cauchy transform of the measure σ_β. The more appropriate notation $\widehat{\sigma_\beta}$ causes space consumption and aesthetic inconveniences. We need to take consecutive products of measures; for example,

$$\langle \sigma_\gamma, \sigma_\alpha, \sigma_\beta \rangle := \langle \sigma_\gamma, \langle \sigma_\alpha, \sigma_\beta \rangle \rangle.$$

Here, we assume not only that $\widehat{\sigma}_\beta \in L_1(\sigma_\alpha)$ but also $\widehat{\langle \sigma_\alpha, \sigma_\beta \rangle} \in L_1(\sigma_\gamma)$ where $\widehat{\langle \sigma_\alpha, \sigma_\beta \rangle}$ denotes the Cauchy transform of $\langle \sigma_\alpha, \sigma_\beta \rangle$. Inductively, one defines products of a finite number of measures.

Definition 1.2.2. *Take a collection* $\Delta_j, \quad j = 1, \ldots, m,$ *of intervals such that, for* $j = 1, \ldots, m-1$

$$\Delta_j \cap \Delta_{j+1} = \emptyset, \qquad \textit{or} \qquad \Delta_j \cap \Delta_{j+1} = \{x_{j,j+1}\},$$

where $x_{j,j+1}$ *is a single point. Let* $(\sigma_1,\ldots,\sigma_m)$ *be a system of measures such that* $Co(\mathrm{supp}(\sigma_j)) = \Delta_j, \sigma_j \in \mathcal{M}(\Delta_j), j = 1,\ldots,m,$ *and*

$$\langle \sigma_j,\ldots,\sigma_k\rangle := \langle \sigma_j, \langle \sigma_{j+1},\ldots,\sigma_k\rangle\rangle \in \mathcal{M}(\Delta_j), \qquad 1 \le j < k \le m. \tag{1.3}$$

When $\Delta_j \cap \Delta_{j+1} = \{x_{j,j+1}\}$ *we also assume that* $x_{j,j+1}$ *is not a mass point of either* σ_j *or* σ_{j+1}*. We say that* $(s_{1,1},\ldots,s_{1,m}) = \mathcal{N}(\sigma_1,\ldots,\sigma_m)$*, where*

$$s_{1,1} = \sigma_1, \quad s_{1,2} = \langle \sigma_1,\sigma_2\rangle, \ldots \quad , s_{1,m} = \langle \sigma_1,\sigma_2,\ldots,\sigma_m\rangle$$

is the Nikishin system of measures generated by $(\sigma_1,\ldots,\sigma_m)$*.*

Initially, E.M. Nikishin in [59] restricted himself to measures with bounded support and no intersection points between consecutive Δ_j. However Definition 1.2.2 includes interesting examples which appear in practice. For example, take $\mathcal{N}(\sigma_1,\sigma_2)$, where

$$d\sigma_1(x) = e^{-x^{\lambda_1}}dx, \quad x \in [0,+\infty), \quad \lambda_1 > 0,$$

$$d\sigma_2(x) = e^{x^{\lambda_2}}dx, \quad x \in (-\infty,0], \quad \lambda_2 > 0,$$

or

$$d\sigma_1(x) = \frac{dx}{\sqrt{x(1-x)}}, \quad x \in [0,1],$$
$$d\sigma_2(x) = dx, \quad x \in [-1,0].$$

These examples with classical weights, and their generalizations, have received considerable attention in brownian motion and random matrix theories (see, [11, 13, 20, 22, 23] and [25]).

In defining a Nikishin systems we follow the approach of [33, Definition 1.2] assuming additionally the existence of all the moments of the generating measures. This is done only for the purpose of simplifying the presentation without affecting too much the generality. However, we wish to point out that the results of this paper have appropriate formulations with the definition given in [33] of a Nikishin system.

Fix $\mathbf{n} \in \mathbb{Z}_+^m$ and consider the type II approximant of the Nikishin system of functions $(\widehat{s}_{1,1},\ldots,\widehat{s}_{1,m})$ relative to $\mathbf{n}$. It is easy to prove that

$$\int x^\nu Q_{\mathbf{n}}(x)ds_{1,j}(x) = 0, \quad \nu = 0,\ldots,n_j-1, \quad j = 1,\ldots,m.$$

All the measures $s_{1,j}$ have the same support; therefore, it is not immediate to conclude that $\deg Q_{\mathbf{n}} = |\mathbf{n}|$. Nevertheless, if we denote

$$s_{j,k} = \langle \sigma_j, \sigma_{j+1}, \dots, \sigma_k \rangle, \qquad j < k, \quad s_{j,j} = \langle \sigma_j \rangle = \sigma_j,$$

the previous orthogonality relations may be rewritten as follows

$$\int (p_1(x) + \sum_{j=2}^{m} p_j(x)\widehat{s}_{2,j}(x))Q_{\mathbf{n}}(x)d\sigma_1(x) = 0, \tag{1.4}$$

where $p_1, \dots, p_m$ are arbitrary polynomials such that $\deg p_j \le n_j - 1, j = 1, \dots, m.$

Definition 1.2.3. *A system of real continuous functions $u_1, \dots, u_m$ defined on an interval Δ is called an AT-system on Δ for the multi-index $\mathbf{n} \in \mathbb{Z}_+^m \setminus \{\mathbf{0}\}$ if for any choice of real polynomials (that is, with real coefficients) $p_1, \dots, p_m, \deg p_j \le n_j - 1,$ the function*

$$\sum_{j=1}^{m} p_j(x)u_j(x)$$

has at most $|\mathbf{n}| - 1$ zeros on Δ. If this is true for all $\mathbf{n} \in \mathbb{Z}_+^m \setminus \{\mathbf{0}\}$ we have an AT system on Δ.

In other words, $u_1, \dots, u_m$ forms an AT-system for $\mathbf{n}$ on Δ when the system of functions

$$(u_1, \dots, x^{n_1-1}u_1, u_2, \dots, x^{n_m-1}u_m)$$

is a Tchebyshev system on Δ of order $|\mathbf{n}| - 1$.

From the properties of Tchebyshev systems (see [46, Theorem 1.1]), it follows that given $x_1, \dots, x_N, N < |\mathbf{n}|$, points in the interior of Δ one can find polynomials $h_1, \dots, h_m$, conveniently, with $\deg h_j \le n_j - 1$, such that $\sum_{j=1}^{m} h_j(x)u_j(x)$ changes sign at $x_1, \dots, x_N$, and has no other points where it changes sign on Δ.

The concept of AT system (algebraic Tchebychev system) was introduced by Nikishin in [59], with the purpose of proving normality of multi-indices for Nikishin systems of measures. In [59], Nikishin stated without proof that the system of functions $(1, \widehat{s}_{2,2}, \dots, \widehat{s}_{2,m})$ forms an AT-system for all multi-indices $\mathbf{n}$ such that $n_1 \ge \cdots \ge n_m$ (he proved it when additionally $n_1 - n_m \le 1$). Due to (1.4) this implies that $\deg Q_{\mathbf{n}} = |\mathbf{n}|$, and all zeros are simple and lie on Δ_1.

The proof of Nikishin's assertion is a consequence of [26, Theorem 4.1]. Actually, Fidalgo and Lagomasino proved in [32, 33] the same results for arbitary multi-indices, thus showing that Nikishin systems are perfect.

Ever since the appearance of [59], a subject of major interest for those involved in simultaneous approximation was the study of the algebraic and analytic properties of the Hermite-Padé approximants with respect to a Nikshin system of measures and their associated type I and type II multiple orthogonal polynomials.

1.3 Main results of the thesis

In Section 2.1, we present mixed type multiple orthogonal polynomials (MTOP) with respect to a matrix measure. Such orthogonality has type I and type II as particular cases. We also define normality of indices and perfectness of matrix measures (see Definition 2.1.1). Making use of the concept of AT system given in Definition 1.2.3 we present in Definition 2.1.8 what we call an AT matrix measure. In Theorem 2.1.10 we prove that AT matrix measures are perfect. This theorem, combined with results known about Angelesco and Nikishin systems, allows to provide a wide class of matrix measures with this desirable property. Section 2.3 contains several examples which appear in practice which illustrate the use of this result. In the rest of Sections 2.1 and 2.2 interlacing properties of the zeros of MTOP and certain associated linear forms are revealed. Section 2.4 reflects the connection between MTOP and mixed type Hermite-Padé approximation. The contents of Chapter 2 was published in [35]. Using a different approach, based on recursion formulas, M. Haneczok and W. Van Assche in [43] also obtained interlacing properties of zeros for some MTOP with respect to matrix measures.

Chapter 3 is devoted to the study of type I Hermite-Padé approximation of a Nikishin system $(s_{1,1},\dots,s_{1,m}) = \mathcal{N}(\sigma_1,\dots,\sigma_m)$. For type II Hermite-Padé approximation of Nikishin systems Markov and Stieltjes type theorems are well known (see, for example [15], [30], [31], [42], [72]). Generally speaking, in these papers it is proved that under appropriate conditions (depending on whether the generating measures have compact support or not) diagonal sequences of type II Hermite-Padé approximants of the system of functions $(\widehat{s}_{1,1},\dots,\widehat{s}_{1,m})$ converge componentwise to the vector function uniformly on compact subsets of $\overline{\mathbb{C}} \setminus \Delta_1$. The goal of Chapter 3 is the proof of Theorem 3.2.1 in

Section 3.2 in which it is shown that type I Hermite-Padé approximants of $(\widehat{s}_{1,1},\ldots,\widehat{s}_{1,m})$ converge to the components of $(\widehat{s}_{m,m},\ldots,\widehat{s}_{m,1})$, where $(s_{m,m},\ldots,s_{m,1})=\mathcal{N}(\sigma_m,\ldots,\sigma_1)$, uniformly on compact subsets of $\overline{\mathbb{C}}\setminus\Delta_m$ and localize the zeros of the type I polynomials in a neighborhood of Δ_m. This phenomenon had not been pointed out before and possibly is the most singular result of the thesis. Section 3.1 contains some auxiliary results needed for the proof of the main result of this chapter. It may be worth singling out Theorem 3.1.5 where a result of independent interest related with Carleman's condition is proved. The contents of this chapter has been submitted for publication (see [55]).

Finally in Chapter 4 we study the convergence of type II Hermite-Padé approximants to a Nikishin system which has been perturbed with rational functions. More precisely, let $(s_{1,1},\ldots,s_{1,m})=\mathcal{N}(\sigma_1,\ldots,\sigma_m)$ be a Nikishin system and Δ_1 be the convex hull of $supp(\sigma_1)$. Let $(r_1,\ldots,r_m)$ be a vector of rational functions with real coefficients such that $r_j(\infty)=0$ and the poles of the r_j are distinct and lie in $\mathbb{C}\setminus(\Delta_1\cup\Delta_m)$, for all $j=1,\ldots,m$. In Theorem 4.2.5 in Section 4.2 we prove the convergence of diagonal sequences of type II Hermite-Padé approximants associated to the system of functions $(f_1,\ldots,f_m)$, where

$$f_j(z)=\int\frac{ds_{1,j}(x)}{z-x}+r_j(z),\qquad j=1,\ldots,m,$$

uniformly on any compact subset of $\overline{\mathbb{C}}\setminus\Delta_1$ which does not contain poles of the r_j. For such perturbed Nikishin systems the only results known appear in [15] where the case when $m=2$ is considered. We study the convergence for any m and recover the main result of [15] when $m=2$. For the proof of Theorem 4.2.5 we need the normality of certain multi-indices for type I Hermite-Padé approximants of systems of the form $(t_0,t_1\widehat{s}_{1,1},\ldots,t_m\widehat{s}_{1,m})$ where the $t_j, j=0,1,\ldots,m$ are polynomials with real coefficients. This is done in Theorem 4.1.1 and Theorem 4.1.2 of Section 4.1. In Section 4.3 we consider the situation when the vector rational function $(r_1,\ldots,r_m)$ has complex coefficients. Here, we restrict the class of Nikishin systems requiring that the intervals Δ_j be bounded and the generating measures be such that $|\sigma_j'|>0$ almost everywhere on Δ_j. In this situation in Theorem 4.3.3 we prove convergence in logarithmic capacity of the corresponding type II Hermite-Padé approximants. The question of uniform convergence for rational perturbations with complex coefficients remains open. The contents of Sections 4.1 and 4.2 were submitted for publication (see [34]).

The results of this thesis have been presented at various international meetings,

- **XVII International Workshop on Wavelets, Differential Equations, Mechanics, Number Theory and Financial Mathematics.** *A Markov type theorem for Hermite-Padé approximation.* Havana, Cuba. February 17, 2014- February 21, 2014. Organizing institutions: Universidad de La Habana, Cuba and Concordia University, Canada.
- **Orthoquad 2014.** *On the convergence of Hermite-Padé approximants.* Tenerife, Spain. January 20, 2014-January 24, 2014. Organizing institution: Universidad de La Laguna, Spain.
- **Segundo Congreso de Jóvenes Investigadores de la Real Sociedad Matemática Española.** *Extensión del teorema de Markov para los aproximantes Hermite-Padé de tipo I de un sistema de Nikishin.* Sevilla, Spain. September 16, 2013-September 20, 2013. Organizing institution: Real Sociedad Matemática Española.
- **12 International Symposium on Orthogonal Polynomials, Special Functions and Applications.** *Convergence of type II Hermite-Padé approximants.* Tunisia. March 24, 2013-March 29, 2013. Organizing institution: SIAM Activity Group on Orthogonal Polynomials and Special Functions
- **Congreso de la Real Sociedad Matemática Española.** *Convergence of type II Hermite-Padé approximants.* Santiago de Compostela, Spain. January 21, 2013-January 25, 2013. Organizing institution: Real Sociedad Matemática Española.

and also in the following seminars

- **Seminario del Departamento de Matemáticas del Grupo de Análisis Funcional. Universidad de Murcia.** *Extensión del teorema de Markov para los aproximantes Hermite-Padé de tipo I de un sistema de Nikishin.* Spain. May 23, 2013.
- **Seminar on Orthogonality, Approximation Theory and Applications. Departamento de Matemáticas, Universidad Carlos III de Madrid.** *Convergence of type II Hermite-Padé approximants.* Spain. March 7, 2013.
- **Seminar Classical Analysis. Department of Mathematics, Katholieke Universiteit Leuven.** *Mixed type multiple orthogonal polynomials: Perfectness and interlacing properties of zeros.* Belgium. May 2, 2012.

- **Seminar on Orthogonality, Approximation Theory and Applications. Departamento de Matemáticas, Universidad Carlos III de Madrid.** *Sobre la perfección de sistemas AT-Nikishin mixtos.* Spain. October 13, 2011

Chapter 2

Multiple orthogonal polynomials

«CONTENTS»

- Definition of mixed type orthogonal polynomials.
- Perfectness of mixed type orthogonal polynomials.
- Interlancing properties of zeros.
- Mixed type Hermite-Pade approximants.

LET $\{Q_n\}$, $n \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, denote the sequence of monic orthogonal polynomials with respect to a measure $s \in \mathcal{M}(\Delta)$. It is well known and easy to verify that for each $n \in \mathbb{Z}_+$, $\deg Q_n = n$, all its zeros are simple and lie in the open convex hull of the support of s, and the zeros Q_n and Q_{n+1} interlace. These properties play an essential role in finding the asymptotic behavior of orthogonal polynomials and applying these results in several areas such as rational approximation, integration by quadratures, birth-and-death processes, Toda lattices, and others. We study a large class of multiple orthogonal polynomials for which similar properties hold.

In recent years, new applications have appeared in which the orthogonal (or bi-orthogonal) system has a more intricate structure. These new areas include, simultaneous rational approximation [29], number theory [12], simultaneous quadrature formulae [19], random matrices [14] and [25], multicomponent Toda latices [2, 3], and non-intersecting Brownian motions [21]. Here again the properties of the zeros of the corresponding orthogonal systems are crucial to determine convergence or to describe the asymptotic behavior of the models under consideration. The purpose of this chapter is to study the properties of the zeros of general bi-orthogonal systems which appear in practice.

In [32] (see also [33]) the authors proved that Nikshin systems are perfect; that is, their associated multiple orthogonal polynomials have maximun degree regardless of the multi-index. In Section 2.1 we give a wide class of matrix measures for which the associated multiple orthogonal polynomials verify this property. In Section 2.2 we prove interlacing properties of the zeros of such multiple orthogonal polynomials and of certain linear forms associated with the perfect systems introduced in Section 2.1. Using a different approach based on recurrence relations, in [43] similar questions were considered. In Section 2.3 we provide some examples of perfect multiple orthogonal polynomials with the interlacing property of their zeros. Finally, in Section 2.4 we introduce mixed type Hermite-Padé approximants and their relation with multiple orthogonal polynomials.

2.1 Perfect matrices of measures

Set

$$\mathbf{S} = \begin{pmatrix} s_{1,1} & \cdots & s_{1,m_1} \\ \vdots & \ddots & \vdots \\ s_{m_2,1} & \cdots & s_{m_2,m_1} \end{pmatrix}, \tag{2.1}$$

where $s_{i,j} \in \mathcal{M}(\Delta)$, $i = 1, \ldots, m_2$, $j = 1, \ldots, m_1$. We write $\mathbf{S} \in \mathcal{M}^{m_2 \times m_1}(\Delta)$. Let us take two multi-indices $\mathbf{n}^\ell = (n_1^\ell, \ldots, n_{m_\ell}^\ell) \in \mathbb{Z}_+^{m_\ell}$, $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, $\ell = 1, 2$. Denote $|\mathbf{n}^1| = n_1^1 + \cdots + n_{m_1}^1$ and $|\mathbf{n}^2| = n_1^2 + \cdots + n_{m_2}^2$. When $|\mathbf{n}^1| = |\mathbf{n}^2| + 1$ there is a vector polynomial $\mathbf{A} = (a_1, \ldots, a_{m_1}) \not\equiv \mathbf{0}$ ($\mathbf{0}$ denotes the null vector) with $\deg a_j < n_j^1$, $j = 1, \ldots, m_1$, ($\deg a_j < 0$ means that $a_j \equiv 0$) such that

$$\sum_{j=1}^{m_1} \int x^\nu a_j(x) ds_{i,j}(x) = 0, \quad \nu = 0, \ldots, n_i^2 - 1, \quad i = 1, \ldots, m_2. \tag{2.2}$$

The existence of $\mathbf{A}$ reduces to solving a homogeneus linear system of $|\mathbf{n}^2|$ equations on the total number of $|\mathbf{n}^1|$ coefficients of the vector polynomial $\mathbf{A}$; therefore, a non-trivial solution is guaranteed.

We call the vector of polynomial $\mathbf{A} = (a_1, \ldots, a_{m_1})$ a mixed type multiple-orthogonal polynomials (MTOP) with respect to $(\mathbf{S}, \mathbf{n}^1, \mathbf{n}^2)$. In what follows, we only consider multi-indices $\mathbf{n} = (\mathbf{n}^1, \mathbf{n}^2)$ satisfying $|\mathbf{n}^1| = |\mathbf{n}^2| + 1$ and write $\mathbf{n} = (\mathbf{n}_1, \mathbf{n}_2) \in (\mathbb{Z}_+^{m_1} \times \mathbb{Z}_+^{m_2})^\star$.

Of course, the components of $\mathbf{A} = \mathbf{A_n}$ depend on $\mathbf{n}$. However, for the time being as long as $\mathbf{n}$ remains fixed, to simplify the notation we will not indicate this dependence.

A formalization of this kind of orthogonality was initiated in [71]. When the matrix $\mathbf{S}$ in (2.1) is a row vector ($m_2 = 1$) the system $(a_1, \ldots, a_{m_1})$ is called type I multi-orthogonal polynomial and if $m_1 = 1$, that is when $\mathbf{S}$ is a column vector, then a_1 is a type II multi-orthogonal polynomial. Standard orthogonality appears when $m_2 = m_1 = 1$.

Definition 2.1.1. *We say that a pair of multi-indices* $\mathbf{n} = (\mathbf{n}^1, \mathbf{n}^2) \in (\mathbb{Z}_+^{m_1} \times \mathbb{Z}_+^{m_2})^\star$ *is normal with respect to a matrix measure* $\mathbf{S} \in \mathcal{M}^{m_2 \times m_1}(\Delta)$ *if for every non zero vector* $\mathbf{A} = (a_1, \ldots, a_{m_1})$ *satisfying* (2.2)*, we have* $\deg a_j = n_j^1 - 1$, $j = 1, \ldots, m_1$. *When every pair of multi-indices* $(\mathbf{n}^1, \mathbf{n}^2) \in (\mathbb{Z}_+^{m_1} \times \mathbb{Z}_+^{m_2})^\star$ *is normal with respect to* $\mathbf{S} \in \mathcal{M}^{m_2 \times m_1}(\Delta)$*, we say that* $\mathbf{S}$ *is perfect.*

Notice that when $\mathbf{S}$ is a row vector $(m_2 = 1)$ $\mathbf{S}$ is perfect if and only if the system of measures $\mathbf{S}$ is type I perfect. Similary, if $\mathbf{S}$ is a column vector $(m_1 = 1)$, $\mathbf{S}$ is perfect if the system of measures $\mathbf{S}^T$ is type II perfect.

Proposicion 2.1.2. *Given* $\mathbf{S} \in \mathcal{M}^{m_2 \times m_1}(\Delta)$ *and a corresponding normal multi-index* $\mathbf{n} = (\mathbf{n}^1, \mathbf{n}^2) \in (\mathbb{Z}_+^{m_1} \times \mathbb{Z}_+^{m_2})^\star$*, the vectors* $\mathbf{A} = (a_1, \ldots, a_{m_1})$ *which satisfy* (2.2)*, are co-linear.*

Proof. Since $|\mathbf{n}^1| > 0$ there is $k \in \{1, \ldots, m_1\}$ where $n_k^1 > 0$. Suppose that there exists another MTOP $\mathbf{A}' = (a_1' \ldots a_{m_1}')$ with respect to the same matrix measure $\mathbf{S}$ and the pair $\mathbf{n} = (\mathbf{n}^1, \mathbf{n}^2) \in (\mathbb{Z}_+^{m_1} \times \mathbb{Z}_+^{m_2})^\star$, such that $\mathbf{A} \not\equiv \alpha \mathbf{A}'$ for every constant $\alpha \in \mathbb{R} \setminus \{0\}$. By the linearity of condition (2.2), for each $\alpha \in \mathbb{R} \setminus \{0\}$ the function $\mathbf{A} - \alpha \mathbf{A}' = \widetilde{\mathbf{A}}$ is also a MTOP for $\mathbf{S}$ and $\mathbf{n}$. However, we can choose α properly to have $\deg \widetilde{a}_k = \deg(a_k - \alpha a_k') < n_k^1 - 1$ which contradicts the fact that $\mathbf{n}$ is normal. $\square$

Take an arbitrary vector polynomial $\mathbf{D} = (d_1, \ldots, d_{m_2})$, where $\deg d_i < n_i^2$, $i = 1, \ldots, m_2$ (when $\deg d_i = -1$ we assume that $d_i \equiv 0$). Relation (2.2) has the following equivalent expression

$$0 = \sum_{j=1}^{m_1} \int d_i(x) a_j(x) ds_{i,j}(x), \qquad i = 1, \ldots, m_2. \tag{2.3}$$

In matrix form this is equivalent to saying that $\mathbf{A} = (a_1, \ldots, a_{m_1})$, $\deg a_j < n_j^1$, $j = 1, \ldots, m_1$ is such that

$$0 = \int \mathbf{D}(x) d\mathbf{S}(x) \mathbf{A}^T(x), \quad d\mathbf{S}(x) = \begin{pmatrix} ds_{1,1}(x) & \cdots & ds_{1,m_1}(x) \\ \vdots & \ddots & \vdots \\ ds_{m_2,1}(x) & \cdots & ds_{m_2,m_1}(x) \end{pmatrix}, \tag{2.4}$$

for any vector polynomial $\mathbf{D} = (d_1, \ldots, d_{m_2})$, $\deg d_k < n_k^2$, $k = 1, \ldots, m_2$. As usual $(\cdot)^T$ denotes taking transpose.

Definition 2.1.3. *Fix a MTOP* $\mathbf{A} = (a_1, \ldots, a_{m_1})$ *with respect to* $(\mathbf{S}, \mathbf{n}^1, \mathbf{n}^2)$*, we say that a vector of polynomial* $\mathbf{B} = (b_1, \ldots, b_{m_2})$ *is dual to* $\mathbf{A}$ *if* $\mathbf{B}$ *is a MTOP with respect to* $(\mathbf{S}^T, \mathbf{n}^2 + \mathbf{e}^2, \mathbf{n}^1 - \mathbf{e}^1)$ *for some* $\mathbf{e}^\ell \in \mathbb{Z}_+^{m_\ell}$ *with* $|\mathbf{e}^\ell| = 1$, $\ell = 1, 2$. *Since* $\mathbf{A}$ *is also dual with respect to* $\mathbf{B}$, *we say that* $\mathbf{A}$ *and* $\mathbf{B}$ *are dual.*

The notion of duality is motivated by ideas discussed in [26], where the authors call the problem of defining type I and type II polynomials dual problems.

Remark 2.1.4. *Notice that a* B *dual to a MTOP* A *is not necessarily uniquely determined. In particular it may depend on the vectors* $\mathbf{e}^1$, $\mathbf{e}^2$ *chosen.*

Proposicion 2.1.5. $\mathrm{S} \in \mathcal{M}^{m_2 \times m_1}(\mathbb{R})$ *is perfect if and only if* $\mathbf{S}^T$ *is perfect.*

Proof. Since $(\mathbf{S}^T)^T = \mathbf{S}$ it is sufficient to prove that if $\mathbf{S}^T$ is perfect then $\mathbf{S}$ is also perfect. Let us assume that $\mathbf{S} \in \mathcal{M}^{m_2 \times m_1}(\mathbb{R})$ is not perfect. Then there exists a multi-index $\mathbf{n} = (\mathbf{n}^1, \mathbf{n}^2) \in (\mathbb{Z}_+^{m_1} \times \mathbb{Z}_+^{m_2})^\star$ and $\mathbf{A} = (a_1, \ldots, a_{m_1})$ which is multiple orthogonal with respect to $(\mathbf{S}, \mathbf{n}^1, \mathbf{n}^2)$ such that for some $k = 1, \ldots, m_1$ $\deg a_{\mathbf{n},k} = n_k^1 - N \leq n_k^1 - 2$.

From (2.4) we have that

$$0 = \int \mathbf{D}(x) d\mathbf{S}(x) \mathbf{A}^T(x)$$

for any vector polynomial $\mathbf{D} = (d_1, \ldots, d_{m_2})$ such that $\deg d_j \leq n_j^2 - 1$, $j = 1, \ldots, m_2$. In particular in place of $\mathbf{D}$ we can put any multiple orthogonal polynomial $\mathbf{B}$ with respect to $\mathbf{S}^T$ and a multi-index of the form $(\tilde{\mathbf{n}}, \mathbf{n}^1 - N e_k^1) \in \mathbb{Z}_+^{m_1} \times \mathbb{Z}_+^{m_2}$ where e_k^1 is the unit vector of $\mathbb{R}^{m_1}$ with 1 in the $k-th$ component and $\tilde{\mathbf{n}} = (\tilde{n}_1, \ldots, \tilde{n}_{m_1})$ satisfies $|\tilde{\mathbf{n}}| = |\mathbf{n}^2| - N + 2$ and $\tilde{n}_j \leq n_j^2$, $j = 1, \ldots, m_2$. Notice that

$$|\tilde{\mathbf{n}}| = \left|\mathbf{n}^2\right| - N + 2 = \left|\mathbf{n}^1\right| - N + 1 = \left|\mathbf{n}^1 - Ne_k^1\right| + 1.$$

Therefore $(\tilde{\mathbf{n}}, \mathbf{n}^1 - Ne_k^1) \in (\mathbb{Z}_+^{m_1} \times \mathbb{Z}_+^{m_2})^*$.

From the definiton of $\mathbf{B}$ as a MTOP, we have that

$$0 = \int \mathbf{C}(x) d\mathbf{S}^T(x) \mathbf{B}^T(x) \tag{2.5}$$

for any vector polynomial $\mathbf{C} = (c_1, \ldots, c_{m_1})$ such that $\deg c_i \leq n_i^1 - 1$, $i = 1, \ldots, m_1$, $i \neq k$, and $\deg c_k \leq n_k^1 - N - 1$. On hte other hands, since $\mathbf{S}^T$ is perfect it follows that

$$0 \neq \int \mathbf{C}(x) d\mathbf{S}^T(x) \mathbf{B}^T(x) \tag{2.6}$$

if $\deg c_i \leq n_i^1 - 1$, $i = 1, \ldots, m_1$, $i \neq k$, and $\deg c_k = n_k^1 - N$.

Substituting $\mathbf{D}$ by $\mathbf{B}$ in (2.1) we get

$$0 = \int \mathbf{B}(x) d\mathbf{S}(x) \mathbf{A}^T(x) = \int \mathbf{A}(x) d\mathbf{S}^T(x) \mathbf{B}^T(x).$$

However the components of $\mathbf{A}$ verify the conditions of the components of $\mathbf{C}$ in (2.6) regarding the degrees. Therefore

$$0 \neq \int \mathbf{A}(x) d\mathbf{S}^T(x) \mathbf{B}^T(x).$$

This contradictions implies that $\mathbf{S}$ must be perfect. □

When $\mathbf{S}$ is a vector we say that it is a vector measure.

Proposicion 2.1.6. *Let $\mathbf{S} = (s_1, \ldots, s_m)$ be an Angelesco vector of measures (see Definition 1.2.1). Then $\mathbf{S}$ is perfect, or what is the same $\mathbf{S}$ is type I perfect.*

Proof. As we showed in Section 1.2 $\mathbf{S}$ is type II perfect, that is, $\mathbf{S}^T$ is perfect. Using Proposition 2.1.5 we have that, $\mathbf{S}$ is also perfect. □

Proposicion 2.1.7. *Let $\mathbf{A} = (a_1, \ldots, a_{m_1})$ be a type I ($m_2 = 1$) multi-orthogonal polynomial relative to an Angelesco system $\mathbf{S}$ and $\mathbf{n} \in \mathbb{Z}_+^{m_1}$. Then, each polynomial a_j, $j = 1, \ldots, m_1$, has exactly $n_j - 1$ simple zeros in Δ_j.*

Proof. By Proposition 2.1.5, $\deg a_j = n_j - 1$, $j = 1, \ldots, m_1$. Let us suppose that for some $k \in \{1, \ldots, m_1\}$, a_k has less than $n_k - 1$ sign changes on Δ_k. Then, there exists a polynomial $p_k, \deg p_k \geq 1$, with real coefficients such, that $a_k(x) = p(x)\widetilde{a}_k(x)$ and p_k has constant sign on Δ_k. The measure $d\tilde{s}_k = p(x)ds_k$ has constant sign on Δ_k. Then, $\widetilde{\mathbf{A}} = (a_1, \ldots, a_{k-1}, \tilde{a}_k, a_{k+1}, \ldots, a_{m_1})$, where $\deg \tilde{a}_k < n_k - 1$, is a MTOP with respect to the Angelesco system $\tilde{\mathbf{S}} = (s_1, \ldots, s_{k-1}, \tilde{s}_k, s_{k+1}, \ldots, s_{m_1})$ and the multi-index $\mathbf{n}$. This contradicts the fact that $\tilde{\mathbf{S}}$ is perfect, which completes the proof. $\square$

One might think of extending Definition 1.2.1 to matrices to obtain an analogous result about perfectness. However, in general, matrices $\mathbf{S} \in \mathcal{M}^{m_2 \times m_1}(\mathbb{R})$ whose entries are supported on disjoint intervals may not be perfect. We give an example.

Fix the pair of multi-indices $\mathbf{n} = (\mathbf{n}^1, \mathbf{n}^2)$ with $\mathbf{n}^1 = (2, 1)$ and $\mathbf{n}^2 = (1, 1)$. We have $|\mathbf{n}^1| = |\mathbf{n}^2| + 1$. Let $m_{[a,b]}$ denote the Lebesgue measure on the interval $[a, b]$. Take the following measure matrix

$$\mathbf{S} = \begin{pmatrix} m_{[-4,-3]} & m_{[-1,-2]} \\ m_{[1,2]} & m_{[3,4]} \end{pmatrix}.$$

Then, the orthogonality relations (2.3) are given by

$$\int_{-4}^{-3} a_1(x)dx + \int_{-1}^{-2} a_2(x)dx = 0$$

$$\int_{1}^{2} a_1(x)dx + \int_{3}^{4} a_2(x)dx = 0,$$

with $\deg a_1 \leq 1$, $\deg a_2 \leq 0$. A solution is $a_1 \equiv 1$ and $a_2 \equiv -1$, where $\deg a_1 = 0$ and $\deg a_2 = 0$. So, this Angelesco type matrix measure is not perfect.

We give examples of perfect matrices of measures using the AT systems of functions defined in Section 1.2 (see Definition 1.2.3). Examples of AT systems are (see [59])

$$(e^{\gamma_1 x}, \ldots, e^{\gamma_m x}), \qquad \gamma_i \neq \gamma_j, \qquad i \neq j, \qquad i, j = 1, \ldots, m, \tag{2.7}$$

with $\Delta = (-\infty, \infty)$, or the binomial functions

$$((1-x)^{\alpha_1}, \ldots, (1-x)^{\alpha_m}), \quad \alpha_i - \alpha_j \notin \mathbb{Z}, \quad i \neq j, \quad i, j = 1, \ldots, m, \tag{2.8}$$

with $\Delta = (-\infty, 1)$.

In [32] (see also [33]), the authors proved that if $(s_{1,1},\ldots,s_{1,m}) = \mathcal{N}(\sigma_1,\ldots,\sigma_m)$, is a Nikishin systems of measures then $(1,\hat{s}_{1,1},\ldots,\hat{s}_{1,m})$ is an AT systems on any interval contained in the complement of the convex hull of $supp(\sigma_1)$.

Using AT systems of weights one can find a wide class of perfect matrix measures.

Definition 2.1.8. *Let* $\mathbf{u} = (u_1,\ldots,u_{m_1})$ *and* $\mathbf{v} = (v_1,\ldots,v_{m_2})$ *be two AT-systems on the same interval* Δ. *Take* $\mu \in \mathcal{M}(\Delta)$. *If the matrix of measures* $\mathbf{S}$, *with differential expression*

$$d\mathbf{S} = \mathbf{v}^T\mathbf{u}\,d\mu = \begin{pmatrix} u_1v_1d\mu & \cdots & u_{m_1}v_1d\mu \\ \vdots & \ddots & \vdots \\ u_1v_{m_2}d\mu & \cdots & u_{m_1}v_{m_2}d\mu \end{pmatrix}$$

belongs to $\mathcal{M}^{m_1\times m_2}(\Delta)$ *(i.e.* $u_jv_kd\mu \in \mathcal{M}(\Delta)$, $j=1,\ldots,m_1$, $k=1,\ldots,m_2$*), we say that* $\mathbf{S}$ *is an AT matrix measure. When* $\mathbf{S}$ *is a vector we say that it is an AT vector measure.*

Let S^1 and S^2 be two given Nikishin systems generated by m_1 and m_2 measures, respectively. $S^1 = (s^1_{1,1},\ldots,s^1_{1,m_1}) = \mathcal{N}(\sigma^1_1,\ldots,\sigma^1_{m_1})$, and $S^2 = (s^2_{1,1},\ldots,s^2_{1,m_2}) = \mathcal{N}(\sigma^2_1,\ldots,\sigma^2_{m_2})$, $\sigma^1_1 = \sigma^2_1$, We underline the fact that both Nikishin systems stem from the same basis measure $\sigma^1_1 = \sigma^2_1$, but there is no other restriction on them. Let us introduce the row vectors

$$\mathbf{v} = (1,\widehat{s}^2_{2,2},\ldots,\widehat{s}^2_{2,m_2}), \qquad \mathbf{u} = (1,\widehat{s}^1_{2,2},\ldots,\widehat{s}^1_{1,m_1})$$

and the $m_2\times m_1$ dimensional measure matrix

$$d\mathbf{S} = \mathbf{v}^T\mathbf{u}d\sigma^1_1,$$

Then $\mathbf{S}$ is an AT matrix measure. In [32, 33] the algebraic and analityc properties of this system are studied.

Theorem 2.1.9. *Let* $\mathbf{A} = (a_1,\ldots,a_{m_1})$ *be a MTOP with respect to an AT matrix measure* $d\mathbf{S} = \mathbf{v}^T\mathbf{u}d\mu$, $\mu \in \mathcal{M}(\Delta)$ *and* $\mathbf{n} = (\mathbf{n}^1,\mathbf{n}^2) \in (\mathbb{Z}^{m_1}_+ \times \mathbb{Z}^{m_2}_+)^\star$. *Then,* $\mathcal{A} = \mathbf{A}\mathbf{u}^T = \sum_{j=1}^{m_1} a_ju_j$ *has* $|\mathbf{n}^2|$ *simple zeros in the interior* $\overset{\circ}{\Delta}$ *of* Δ *(the interior with the Eucliedean topology of* $\mathbb{R}$*).*

Proof. Combining linearly the multi-orthogonal relations in (2.3), we obtain that

$$\int (p_1v_1 + \cdots + p_{m_2}v_{m_2})\mathcal{A}(x)d\mu(x) = 0, \tag{2.9}$$

for every vector polynomial $(p_1, \ldots, p_{m_2})$ with $\deg p_i(x) \le n_i^2 - 1$. Let us assume that the linear form $\mathcal{A} = a_1 u_1 + \cdots + a_{m_1} u_{m_1}$ changes sign at the points $\{x_1, \ldots, x_d\} \in \overset{\circ}{\Delta}$, $d < |\mathbf{n}_2|$. Since $(v_1, \ldots, v_{m_2})$ is an AT system, by Theorem 2.1.3 of [46], we can construct a linear form

$$\mathcal{P}(x) = q_1(x)v_1(x) + \cdots + q_{m_2}(x)v_{m_2}(x),$$

with $\deg q_i \le n_i^2 - 1$, which changes sign at the same points as $\mathcal{A}$ in Δ i.e. $\{x_1, \ldots, x_d\}$ and has no other sign change on Δ. Therefore, the product $\mathcal{A}(x)\mathcal{P}(x)$ does not change sign in the interior of Δ. Then

$$\int \mathcal{P}(x)\mathcal{A}_{\mathbf{n}}(x)d\mu(x) \neq 0.$$

This contradicts (2.9). Thus, $d \ge |\mathbf{n}_2|$.

On the other hand, $(u_1, \ldots, u_{m_1})$ forms an AT system on Δ, hence $\mathcal{A}$ has at most $|\mathbf{n}^1| - 1 = |\mathbf{n}^2|$ zeros in Δ. Therefore, d must equal $|\mathbf{n}^2|$ which completes the proof. □

Theorem 2.1.10. *Let* $\mathbf{A} = (a_1, \ldots, a_{m_1})$ *be a MTOP with respect to an AT matrix measure* $\mathbf{S}$ *and* $\mathbf{n} = (\mathbf{n}^1, \mathbf{n}^2) \in (\mathbb{Z}_+^{m_1} \times \mathbb{Z}_+^{m_2})^\star$. *Then,* $\deg a_j = n_j^1 - 1$ *for every* $j = 1, \ldots, m_1$. *Moreover,* $\mathbf{S}$ *is perfect*

Proof. The linear form $\mathcal{A} = a_1 u_1 + \cdots + a_{m_1} u_{m_1}$ has $|\mathbf{n}^1| - 1$ simple zeros in $\overset{\circ}{\Delta}$. On the other hand, since $(u_1, \ldots, u_{m_1})$ is an AT-system for the multi-index $\mathbf{n}_1 - \mathbf{e}_k^1$, if we suppose that there exists k, $0 \le k \le m_1$, such that $\deg a_k < n_k^1 - 1$ then $\mathcal{A} = a_1 u_1 + \cdots + a_{m_1} u_{m_1}$ would have at most $|\mathbf{n}^1| - 2$ zeros, which is imposible. Since this is true for any $(\mathbf{n}^1, \mathbf{n}^2) \in (\mathbb{Z}_+^{m_1} \times \mathbb{Z}_+^{m_2})^\star$ this means that $\mathbf{S}$ is perfect. □

Definition 2.1.11. *Fix* m_1 *disjoint intervals* $\Delta_j \subset \mathbb{R}$, $j = 1, \ldots, m_1$. *Let* $\mathbf{s} = (s_1, \ldots, s_{m_1})$ *be an Angelesco vector measure where* $s_j \in \mathcal{M}(\Delta_j)$ *and let* $\mathbf{v} = (v_1, \ldots, v_{m_2})$ *be an AT-system of weights on Co*$(\cup_{j=1}^{m_1} \Delta_j)$. *If the matrix* $\mathbf{S}$ *with differential form*

$$d\mathbf{S}(x) = \mathbf{v}^T(x)d\mathbf{s}(x) = \begin{pmatrix} v_1(x)ds_1(x) & \cdots & v_1(x)ds_{m_1}(x) \\ \vdots & \ddots & \vdots \\ v_{m_2}(x)ds_1(x) & \cdots & v_{m_2}(x)ds_{m_1}(x) \end{pmatrix}$$

belongs to $\mathcal{M}^{m_2 \times m_1}(\mathbb{R})$, *we say that* $\mathbf{S}$ *(and* $\mathbf{S}^T$*) is an AT-Angelesco matrix measure.*

Theorem 2.1.12. *Let $d\mathrm{S} = \mathbf{v}^T d\mathbf{s}$ be an AT-Angelesco matrix measure where the vector of measures $\mathbf{s} = (s_1, \ldots, s_{m_1})$, is such that $s_j \in \mathcal{M}(\Delta_j)$, $j = 1, \ldots, m_1$, and $\mathbf{v} = (v_1, \ldots, v_{m_2})$ is an AT systems of weights on $\Delta = Co(\cup_{j=1}^{m_1} \Delta_j)$. Let $\mathbf{A} = (a_1, \ldots, a_{m_1})$ be a MTOP corresponding to S and $\mathbf{n} = (\mathbf{n}^1, \mathbf{n}^2) \in (\mathbb{Z}^{m_1} \times \mathbb{Z}^{m_2})^\star$. Then, for each $j = 1, \ldots, m_1$, the polynomial a_j has $n_j^1 - 1$ simple zeros in $\overset{\circ}{\Delta}_j$.*

Proof. Let $\mathrm{S} \in \mathcal{M}^{m_1 \times m_2}(\mathbb{R})$ be an AT-Angelesco matrix measure. Fix a pair of multi-indices $\mathbf{n} = (\mathbf{n}^1, \mathbf{n}^2)$ and set $\mathbf{A} = (a_1, \ldots, a_{m_1})$ a MTOP corresponding to S and $\mathbf{n}$. That is

$$0 = \sum_{j=1}^{m_1} \int x^\nu a_j(x) v_i(x) ds_j(x), \quad \nu = 0, \ldots, n_i^2 - 1, \quad i = 1, \ldots, m_2,$$

where $\deg a_j \leq n_j - 1$, $j = 1, \ldots, m_1$. For each $j = 1, \ldots, m_1$, let κ_j be the number of sign changes of a_j in the interior of Δ_j. Every linear form

$$\mathcal{P}(x) = \sum_{i=1}^{m_2} p_i(x) v_i(x), \quad \text{with} \quad \deg p_i \leq n_i^2 - 1, \quad i = 1, \ldots, m_2,$$

satisfies

$$0 = \sum_{j=1}^{m_1} \int \mathcal{P}(x) a_j(x) ds_j(x). \tag{2.10}$$

Suppose that $\sum_{j=1}^{m_1} \kappa_j < |\mathbf{n}^1| - m_1$. Let us construct $\mathcal{P}$ with simple zeros at the $\sum_{j=1}^{m_1} \kappa_j$ points where the a_j, $j = 1, \ldots, m_1$, changes sign and place other simple zeros in $\mathrm{Co}(\Delta_j \cup \Delta_{j+1}) \setminus (\Delta_j \cup \Delta_{j+1})$, $j = 1, \ldots, m_1 - 1$, in such a way that the products $\mathcal{P} a_j$ are nonnegative in Δ_j, $j = 1, \ldots, m_1$, respectively. However, this contradicts the equality in (2.10). Thus $\kappa_j = n_j - 1$ for each $j = 1, \ldots, m_2$. $\square$

Remark 2.1.13. *Theorem 2.1.12 implies that S is perfect and using Proposition 2.1.5 we have that S^T is also perfect.*

Theorem 2.1.14. *Let $d\mathrm{S}^T = \mathbf{u}^T d\mathbf{s}$ be an AT-Angelesco matrix measure where the vector of measures $\mathbf{s} = (s_1, \ldots, s_{m_2})$, is such that $s_i \in \mathcal{M}(\Delta_i)$, $i = 1, \ldots, m_2$, and $\mathbf{u} = (u_1, \ldots, u_{m_1})$ is an AT system of weights on $\Delta = Co(\cup_{i=1}^{m_2} \Delta_i)$. Let $\mathbf{A} = (a_1, \ldots, a_{m_1})$ be a MTOP corresponding to S and $\mathbf{n} = (\mathbf{n}^1, \mathbf{n}^2) \in (\mathbb{Z}_+^{m_1} \times \mathbb{Z}_+^{m_2})^\star$. Then the linear form $\mathcal{A}(x) = \mathbf{A}(x) \mathbf{u}^T(x)$ has $|\mathbf{n}^1| - 1$ simple zeros in $\overset{\circ}{D} = \cup_{i=1}^{m_2} \overset{\circ}{\Delta}_i$. Moreover, $\deg a_j = n_j^1 - 1$, $j = 1, \ldots, m_1$, which implies that S is perfect.*

Proof. Rewrite the orthogonality relations in (2.3) for the current case as

$$0 = \int x^\nu \mathcal{A}(x) ds_i(x), \quad \nu = 0, \ldots, n_i^2 - 1, \quad i = 1, \ldots, m_2. \tag{2.11}$$

Suppose that there exists $k \in \{1, \ldots, m_2\}$ such that $\mathcal{A}$ changes its sign κ times in Δ_k where $\kappa < n_k^2$. By the equalities (2.11) for $j = k$, we obtain that

$$0 = \int p(x)\mathcal{A}(x) ds_k(x), \tag{2.12}$$

where p is an arbitrary polynomial of degree $\leq n_k^2 - 1$. Let us choose p with κ simple zeros at the same points where $\mathcal{A}$ changes sign in the interior of Δ_k. So the product $p(x)\mathcal{A}(x)$ does not change sign in Δ_k. This contradicts (2.12). Therefore, $\mathcal{A}$ has at least $|\mathbf{n}^1| - 1$ zeros in $\overset{\circ}{D}$ and as $\mathbf{u}$ is an AT system, we conclude that $\mathcal{A}$ has exactly $|\mathbf{n}^1| - 1$ zeros in $\overset{\circ}{D}$. This fact and the AT condition imply that all the polynomials a_j, $j = 1, \ldots, m_1$, have the greatest possible degrees. $\square$

2.2 Interlacing properties of zeros

We are going to prove some interlacing properties of zeros of sequences of mixed type multi-orthogonal polynomials. To this end, we need to state first the following lemma.

Lemma 2.2.1. *Let f and g be two real functions defined on an interval $\Delta \subset \mathbb{R}$ with continuous derivatives (in short $\mathcal{C}^1(\Delta)$) such that the Wronskian of f and g satisfies*

$$W(f, g; x) = det \begin{pmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{pmatrix} \neq 0, \tag{2.13}$$

for any $x \in \Delta$. Then the zeros of f and g in Δ are simple and interlace.

Proof. Suppose that f has a zero of multiplicity greater than one at a point $x_0 \in \Delta$, then f' also vanishes at this point as well as the determinant (2.13) against our assumption. The same goes with g. So the zeros of the functions f and g in Δ are simple. Let x_1 and x_2 be two consecutive zeros of g. Since $f, g \in \mathcal{C}^1(\Delta)$ then $W(f, g; x)$ is a continuous function of $x \in \Delta$. This leads to

$$\text{sgn}(f(x_1)g'(x_1)) = \text{sgn}(f(x_2)g'(x_2)),$$

where sgn denotes the sign function. So, f changes its sign between x_1 and x_2. This implies that it has at least a zero in the interval $[x_1, x_2]$. Let us suppose that f has two different zeros in the interior of (x_1, x_2). Proceeding analogously, we can prove that there is a zero of g between two consecutive zeros of f in $[x_1, x_2]$. This contradicts the fact that x_1 and x_2 were consecutive zeros of g and completes the proof. $\square$

Theorem 2.2.2. *Fix a pair of multi-indices* $\mathbf{n} = (\mathbf{n}^1; \mathbf{n}^2) \in (\mathbb{Z}_+^{m_1} \times \mathbb{Z}_+^{m_2})^\star$ *and* $(j,k) \in \{1,\ldots,m_1\} \times \{1,\ldots,m_2\}$. *Let* $\mathbf{n}_+(j,k) = (\mathbf{n}^1 + \mathbf{e}_j^1, \mathbf{n}^2 + \mathbf{e}_k^2)$ *denote another pair of multi-indices. Let* $\mathbf{A_n}$ *and* $\mathbf{A}_{\mathbf{n}_+(j,k)}$ *be two MTOP with respect to an AT matrix measure* $\mathbf{S} \in \mathcal{M}^{m_2\times m_1}(\Delta)$, *and* $\mathbf{n}$ *and* $\mathbf{n}_+(j,k)$, *respectively. Suppose that* $d\mathbf{S} = \mathbf{v}^T\mathbf{u}d\mu$ *where* $\mathbf{u}$ *is a system of continuously differentiable functions on* Δ *(*$\mathbf{u} \in \mathcal{C}^{1,m_1}(\Delta)$*). Then, the linear forms* $\mathcal{A}_\mathbf{n} = \mathbf{A_n}\mathbf{u}^T$ *and* $\mathcal{A}_{\mathbf{n}_+(j,k)} = \mathbf{A}_{\mathbf{n}_+(j,k)}\mathbf{u}^T$ *interlace their zeros.*

Proof. Let $\mathcal{P}(x)$ denote the linear form

$$\mathcal{P}(x) = A\mathcal{A}_\mathbf{n}(x) + B\mathcal{A}_{\mathbf{n}_+(j,k)}(x), \quad |A| + |B| > 0, \quad A, B \in \mathbb{R}.$$

From (2.3) we obtain the following orthogonality relations

$$\int (p_1 v_1 + \cdots + p_{m_2} v_{m_2})(x)\mathcal{P}(x)d\mu(x) = 0,$$

where $\deg p_i \leq n_i^2 - 1$. Therefore the linear form $\mathcal{P}$ has at least $|\mathbf{n}^1| - 1$ simple zeros in $\overset{\circ}{\Delta}$. On the other hand, $\mathcal{P}$ is a linear form of the AT system $(u_1,\ldots,u_{m_1})$ for the multi-index $\mathbf{n}^1 + \mathbf{e}_j$, so $\mathcal{P}$ has at most $|\mathbf{n}^1|$ zeros in Δ. This implies that the zeros of $\mathcal{P}$ in Δ are simple.

Assume that there exists $y \in \mathrm{Co}(\mathrm{supp}(\mu))$ such that

$$\mathcal{A}_\mathbf{n}(y) = \mathcal{A}_{\mathbf{n}_+(j,k)}(y) = 0.$$

Observe that y is a simple zero for $\mathcal{A}_\mathbf{n}$ and $\mathcal{A}_{\mathbf{n}_+(j,k)}$, then

$$\mathcal{A}'_\mathbf{n}(y) \neq 0 \quad \text{and} \quad \mathcal{A}'_{\mathbf{n}_+(j,k)}(y) \neq 0.$$

Take $\mathcal{P}$ as follows

$$\mathcal{P}(x) = \mathcal{A}_\mathbf{n}(x) - \frac{\mathcal{A}'_\mathbf{n}(y)}{\mathcal{A}'_{\mathbf{n}_+(j,k)}(y)}\mathcal{A}_{\mathbf{n}_+(j,k)}(x),$$

which has at y a double zero. Hence, such a y does not exist. Consequently, $\mathcal{A}_\mathbf{n}$ and $\mathcal{A}_{\mathbf{n}_+(j,k)}$ do not have common zeros in $\mathrm{Co}(\mathrm{supp}(\mu))$.

Now consider the following linear form

$$\mathcal{P}(x)=\mathcal{A}_{\mathbf{n}_+(j,k)}(y)\mathcal{A}_{\mathbf{n}}(x)-\mathcal{A}_{\mathbf{n}}(y)\mathcal{A}_{\mathbf{n}_+(j,k)}(x),$$

that satisfies $\mathcal{P}(y)=0$, so $\mathcal{P}'(y)\neq 0$, i. e.

$$\det\begin{pmatrix}\mathcal{A}_{\mathbf{n}}(y) & \mathcal{A}_{\mathbf{n}_+(j,k)}(y)\\ \mathcal{A}'_{\mathbf{n}}(y) & \mathcal{A}'_{\mathbf{n}_+(j,k)}(y)\end{pmatrix}\neq 0.$$

Using Lemma 2.2.1 the result follows. □

Theorem 2.2.3. *Let* $\mathbf{n}=(\mathbf{n}^1;\mathbf{n}^2)\in(\mathbb{Z}_+^{m_1}\times\mathbb{Z}_+^{m_2})^\star$ *and* $\mathbf{n}_+(j,k)=(\mathbf{n}^1+\mathbf{e}_j^1,\mathbf{n}^2+\mathbf{e}_k^2)$ *be given as in Theorem 2.2.2. Let* $\mathbf{A}_{\mathbf{n}}$ *and* $\mathbf{A}_{\mathbf{n}_+(j,k)}$ *be MTOP with respect to a matrix measure* $\mathbf{S}\in\mathcal{M}^{m_2\times m_1}(\mathbb{R})$, $\mathbf{n}$ *and* $\mathbf{n}_+(j,k)$*, respectively. Let us suppose that* $d\mathbf{S}^T=\mathbf{u}^T d\mathbf{s}$ *is an AT-Angelesco matrix measure with* $\mathbf{u}\in\mathcal{C}^{1,m_2}(\Delta)$*. Then* $\mathcal{A}_{\mathbf{n}}=\mathbf{A}_{\mathbf{n}}\mathbf{u}^T$ *and* $\mathcal{A}_{\mathbf{n}_+(j,k)}=\mathbf{A}_{\mathbf{n}_+(j,k)}\mathbf{u}^T$ *interlace their zeros.*

Proof. Let $\mathcal{P}(x)$ denote the linear form

$$\mathcal{P}(x)=A\mathcal{A}_{\mathbf{n}}(x)+B\mathcal{A}_{\mathbf{n}_+(j,k)}(x),\quad |A|+|B|>0,\quad A,B\in\mathbb{R}.$$

From (2.3) we obtain the following orthogonality relations

$$\int x^\nu\mathcal{P}(x)ds_i(x)=0,\quad \nu=0,\ldots,n_i^2-1,\quad i=1,\ldots,m_2.$$

Then, for each $i\in\{1,\ldots,m_2\}$ the linear form $\mathcal{P}$ has at least n_i^2 simple zeros in $\mathring{\Delta}_i$. So $\mathcal{P}$ has $|\mathbf{n}^2|$ simple zeros in $\mathring{\Delta}=\cup_{i=1}^{m_2}\mathring{\Delta}_2$. On the other hand, $(u_1,\ldots,u_{m_1})$ is an AT system, so $\mathcal{P}$ has at most $|\mathbf{n}^2|+1$ zeros in Δ. This implies that the zeros of $\mathcal{P}$ in Δ are simple. Following the same procedure as in the proof of Theorem 2.2.2, we complete the proof. □

Theorem 2.2.4. *Let* $\mathbf{n}=(\mathbf{n}^1;\mathbf{n}^2)\in(\mathbb{Z}_+^{m_1}\times\mathbb{Z}_+^{m_2})^\star$ *and* $\mathbf{n}_+(j,k)=(\mathbf{n}^1+\mathbf{e}_j,\mathbf{n}^2+\mathbf{e}_k)$ *be given. Let* $\mathbf{A}_{\mathbf{n}}=(a_{\mathbf{n},1},\ldots,a_{\mathbf{n},m_1})$ *and* $\mathbf{A}_{\mathbf{n}_+(j,k)}=(a_{\mathbf{n}_+(j,k),1},\ldots,a_{\mathbf{n}_+(j,k),m_1})$ *be MTOP with respect to a matrix measure* $\mathbf{S}\in\mathcal{M}^{m_2\times m_1}(\mathbb{R})$, $\mathbf{n}=(\mathbf{n}^1,\mathbf{n}^2)$ *and* $\mathbf{n}_+(j,k)=(\mathbf{n}^1+\mathbf{e}_j,\mathbf{n}^2+\mathbf{e}_k)$*, respectively. Let us suppose that* $d\mathbf{S}=\mathbf{v}^T d\mathbf{s}$ *is an AT-Angelesco matrix measure. Then for each* $i=1,\ldots,m_1$, $a_{\mathbf{n},i}$ *and* $a_{\mathbf{n}_+(j,k),i}$ *interlace their zeros.*

Proof. For each $j = 1, \dots, m_1$, let $P_j(x)$ denote the polynomial

$$P_j(x) = Aa_{\mathbf{n},j}(x) + Ba_{\mathbf{n}_+(r,k),j}(x), \quad |A| + |B| > 0, \quad A, B \in \mathbb{R}.$$

For every linear form $\mathcal{L} = p_1 v_1 + \cdots + p_{m_2} v_{m_2}$, where $\deg p_j \leq n_j^2 - 1$, $j = 1, \dots, m_2$, using (2.3) we obtain that

$$\sum_{j=1}^{m_1} \int \mathcal{L}(x) P_j(x) ds_j(x) = 0, \tag{2.14}$$

which are the orthogonality relations stated in (2.10). Since P_j has at least $n_j^1 - 1$ simple zeros in Δ_j where $\deg P_j \leq n_j - 1$, $j = 1, \dots, r-1, r+1, \dots, m_1$, and $\deg P_r \leq n_r$, then all its zeros are simple. Once we have arrived here, we can follow step by step the proof of Theorems 2.2.2 or 2.2.3 above to conclude. $\square$

2.3 Some examples

In this section, we give some examples of AT matrices of measures. As follows from Section 2.1 and Section 2.2, these matrices are perfect and their corresponding MTOP interlace their zeros.

2.3.1 Mixed type multiple Hermite polynomials

Multiple Hermite polynomials are orthogonal with respect to measures $(\mu_1, \dots, \mu_r)$ which are given by $d\mu_j(x) = e^{-x^2 + c_j x} dx$ on $(-\infty, \infty)$ where $c_i \neq c_j$ for $i \neq j$. These weights make up an AT system. We take two AT systems as follows:

$$\mathbf{u} = (e^{-x^2 + c_1 x}, \dots, e^{-x^2 + c_{m_1} x}), \quad \mathbf{v} = (e^{-x^2 + d_1 x}, \dots, e^{-x^2 + d_{m_2} x}).$$

where $c_1 < \ldots < c_{m_1}$, $d_1 < \ldots < d_{m_2}$. Then we can construct an AT matrix measure as follows

$$d\mathbf{S} = \mathbf{v}^T \mathbf{u}\, dx = \begin{pmatrix} e^{-2x^2 + (c_1 + d_1)x} dx & \cdots & e^{-2x^2 + (c_{m_1} + d_1)x} dx \\ \vdots & \ddots & \vdots \\ e^{-2x^2 + (c_1 + d_{m_2})x} dx & \cdots & e^{-2x^2 + (c_{m_1} + d_{m_2})x} dx \end{pmatrix}.$$

From Theorem 2.1.10 this matrix of measures is perfect and from Theorem 2.2.2 their corresponding MTOP interlace their zeros.

Note that changing variables we obtain the matrix

$$d\mathbf{S} = \begin{pmatrix} e^{-t^2+f_{1,1}t}\frac{dt}{\sqrt{2}} & \cdots & e^{-t^2+f_{m_1,1}t}\frac{dt}{\sqrt{2}} \\ \vdots & \ddots & \vdots \\ e^{-t^2+f_{1,m_2}t}\frac{dt}{\sqrt{2}} & \cdots & e^{-t^2+f_{m_1,m_2}t}\frac{dt}{\sqrt{2}} \end{pmatrix},$$

where every row and every column contains an AT system of weights whose multiple orthogonal polynomials are the multiple Hermite polynomials. In [21] and [24], mixed type Hermite polynomials appear in non-intersecting Brownian motion models.

2.3.2 Mixed type multiple Laguerre polynomials of the second kind

Multiple Laguerre polynomials of the second kind are orthogonal with respect to measures $(\mu_1, \ldots, \mu_r)$ which are given by $d\mu_j(x) = x^\alpha e^{-c_j x} dx$ on $[0, \infty)$ where $\alpha > -1$, $c_j > 0$ and $c_i \neq c_j$ whenever $i \neq j$. These weights form an AT system. We take

$$\mathbf{u} = (x^\alpha e^{-c_1 x}, \ldots, x^\alpha e^{-c_{m_1} x}), \quad \alpha > -1,\ c_j > 0,\ c_i \neq c_j,\ i \neq j,$$

and

$$\mathbf{v} = (x^\beta e^{-d_1 x}, \ldots, x^\beta e^{-d_{m_2} x}), \quad \beta > -1,\ d_j > 0,\ d_i \neq d_j,\ i \neq j.$$

Then we construct the following AT matrix measure

$$d\mathbf{S} = \mathbf{v}^T \mathbf{u}\, dx = \begin{pmatrix} x^{\alpha+\beta} e^{-(c_1+d_1)x} dx & \cdots & x^{\alpha+\beta} e^{-(c_{m_1}+d_1)x} dx \\ \vdots & \ddots & \vdots \\ x^{\alpha+\beta} e^{-(c_1+d_{m_2})x} dx & \cdots & x^{\alpha+\beta} e^{-(c_{m_1}+d_{m_2})x} dx \end{pmatrix}.$$

Note that this matrix can be written as

$$d\mathbf{S} = \mathbf{v}^T \mathbf{u}\, dx = \begin{pmatrix} x^\gamma e^{-f_{1,1}x} dx & \cdots & x^\gamma e^{-f_{m_1,1}x} dx \\ \vdots & \ddots & \vdots \\ x^\gamma e^{-f_{1,m_2}x} dx & \cdots & x^\gamma e^{-f_{m_1,m_2}x} dx \end{pmatrix},$$

where every row and column contains an AT system of weights whose multiple orthogonal polynomials are the multiple Laguerre polynomials of the second kind.

From Theorem 2.1.10 this matrix of measures is perfect and from Theorem 2.2.2 their corresponding MTOP interlace their zeros.

2.3.3 Mixed type multiple Laguerre polynomials of the first-second kind

Multiple Laguerre polynomials of the first kind are orthogonal with respect to AT systems of weights $(\mu_1, \dots, \mu_r)$ with $d\mu_j(x) = x^{\alpha_j} e^{-x} dx$ on $[0, \infty)$ where $\alpha_i - \alpha_j \notin \mathbb{Z}$ whenever $i \neq j$, $\alpha_j > -1$.

Now we combine multiple Laguerre polynomials of the first and the second kind. Set

$$\mathbf{u} = (x^{\alpha_1} e^{-x}, \dots, x^{\alpha_{m_1}} e^{-x}), \quad \alpha_i - \alpha_j \notin \mathbb{Z},\, i \neq j,\, \alpha_j > -1,\, j = 1, \dots, m_1,$$

and

$$\mathbf{v} = (x^{\beta} e^{-d_1 x}, \dots, x^{\beta} e^{-d_{m_2} x}), \quad d_i \neq d_j,\, i \neq j.$$

Both vectors are defined on $[0, \infty)$. Then, we construct

$$d\mathbf{S} = \mathbf{v}^T \mathbf{u}\, dx = \begin{pmatrix} x^{\beta+\alpha_1} e^{-(1+d_1)x} dx & \cdots & x^{\beta+\alpha_{m_1}} e^{-(1+d_1)x} \\ \vdots & \ddots & \vdots \\ x^{\beta+\alpha_1} e^{-(1+d_{m_2})x} dx & \cdots & x^{\beta+\alpha_{m_1}} e^{-(1+d_{m_2})x} dx \end{pmatrix},$$

that we know is perfect and their corresponding MTOP interlace their zeros.

2.4 Mixed type Hermite-Padé approximants

The mixed type multiple-orthogonal polynomials have a close link with the theory of Hermite-Padé approximants, In [71], Sorokin introduced the following construction. Let $\mathbf{S} \in \mathcal{M}^{m_2 \times m_1}(\Delta)$.

$$\mathbf{S} = \begin{pmatrix} s_{1,1} & \cdots & s_{1,m_1} \\ \vdots & \ddots & \vdots \\ s_{m_2,1} & \cdots & s_{m_2,m_1} \end{pmatrix}$$

Define the matrix Markov type function

$$\widehat{\mathbf{S}}(z) = \int \frac{d\mathbf{S}(\mathbf{x})}{z - x}$$

understanding that integration is carried out entry by entry on the matrix $\mathbf{S}$.

Fix a multi-index $\mathbf{n} = (\mathbf{n}_1; \mathbf{n}_2) \in \mathbb{Z}_+^{m_1} \times \mathbb{Z}_+^{m_2}$, such that $|\mathbf{n}_1| = |\mathbf{n}_2| + 1$. We denote $\mathbf{n}_i = (n_{i,1}, \ldots, n_{i,m_i}), i = 1, 2$. There exists a vector polynomial $\mathbf{A_n} = (a_{\mathbf{n},1}, \ldots, a_{\mathbf{n},m_1})$, such that

i) $\mathbf{A_n} \not\equiv \mathbf{0}, \deg a_{\mathbf{n},j} \leq n_{1,j} - 1, j = 1, \ldots, m_1, \quad (\deg a_{\mathbf{n},j} = -1$ means $a_{\mathbf{n},j} \equiv 0)$

ii) $(\widehat{\mathbf{S}}\mathbf{A}_\mathbf{n}^T - \mathbf{B}_\mathbf{n}^T)(z) = (\mathcal{O}(1/z^{n_{2,1}+1}), \ldots, \mathcal{O}(1/z^{n_{2,m_2}+1})^T =: \mathcal{O}(1/z^{\mathbf{n}_2+1}), z \to \infty.$

for some m_2 dimensional vector polynomial $\mathbf{B_n}$ (the super-index T means taking transpose and $\mathbf{0}$ denotes the zero vector). Finding $\mathbf{A_n}$ reduces to solving a linear homogeneous system of $|\mathbf{n}_2|$ equations determined by the conditions *ii)* on $|\mathbf{n}_1|$ unknowns (the total number of coefficients of the polynomials $a_{\mathbf{n},j}, j = 1, \ldots, m_1$). Since $|\mathbf{n}_2| + 1 = |\mathbf{n}_1|$ a non trivial solution exists.

Definition 2.4.1. *A non zero vector $\mathbf{A_n}$ satisfying i)-ii) is called mixed type Hermite-Padé approximant relative to $\widehat{\mathbf{S}}$ and $\mathbf{n} \in (\mathbb{Z}_+^{m_1} \times \mathbb{Z}_+^{m_2})^\star$.*

This construction has as particular cases type I $(m_2 = 1)$, type II $(m_1 = 1)$ Hermite-Padé approximants and classical Padé approximants when $m_2 = 1$ and $m_1 = 1$.

Lemma 2.4.2. *Let $\widehat{\mathbf{S}}$ and $\mathbf{n} \in (\mathbb{Z}_+^{m_1} \times \mathbb{Z}_+^{m_2})^\star$ be given. Then mixed type Hermite-Padé approximants relative to $\widehat{\mathbf{S}}$ and $\mathbf{n}$ coincide with mixed type multiple-orthogonal polynomials with respect to $\mathbf{S}$ and $\mathbf{n}$. That is $\mathbf{A_n}$ satisfies*

$$0 = \int \mathbf{D_n}(x) d\mathbf{S}(x) \mathbf{A}_\mathbf{n}^T(x), \quad d\mathbf{S}(x) = \begin{pmatrix} ds_{1,1}(x) & \cdots & ds_{1,m_1}(x) \\ \vdots & \ddots & \vdots \\ ds_{m_2,1}(x) & \cdots & ds_{m_2,m_1}(x) \end{pmatrix}, \tag{2.15}$$

where $\mathbf{D_n} = (d_{\mathbf{n},1} \ldots, d_{\mathbf{n},m_2})$ is an arbitrary vector polynomial such that $\deg d_{\mathbf{n},i} < n_i^2$, $i = 1, \ldots, m_2$ (when $\deg d_{\mathbf{n},i} = -1$ we assume that $d_{\mathbf{n},i} \equiv 0$).

This result may be proved as Lemma 3.1.2 below without any substantial change, so we omit it.

Chapter 3

Convergence of type I Hermite-Padé approximants

«CONTENTS»

- Definition of Hausdorff content and Gonchar´s Lemma.
- Uniform convergence of type I Hermite-Padé approximants.
- Some consequences of convergence of type I Hermite-Padé approximants.

PADÉ approximation has two natural extensions to vector rational approximation through the so called type I and type II Hermite-Padé approximants. The convergence properties of type II Hermite-Padé approximants have been studied. For such approximants Markov and Stieltjes type theorems are available. In this chapter, we provide Markov and Stieltjes type theorems on the convergence of type I Hermite-Padé approximants for Nikishin systems of functions. As a consequence of this result we can detect the location of the zeros of the type I Hermite-Padé approximants and we obtain some interlacing properties of their zeros.

This chapter is organized as follows. In Section 3.1 we present some notions and auxiliary results. Section 3.2 contains the proof of the main theorem of this chapter and some extensions of it, estimates of the rate of convergence for the case when Δ_m or Δ_{m-1} is bounded, and applications to other simultaneous approximation schemes.

3.1 Preliminary notions

Recall that given a system of finite Borel measures $S = (s_1, \ldots, s_m)$ with constant sign and a multi-index $\mathbf{n} = (n_1, \ldots, n_m) \in \mathbb{Z}_+^m \setminus \{\mathbf{0}\}$, $|\mathbf{n}| = n_1 + \cdots + n_m$, where $\mathbb{Z}_+$ denotes the set of non-negative integers and $\mathbf{0}$ the m-dimensional zero vector, their exist polynomials $a_{\mathbf{n},j}, j = 0, \ldots, m$, not all identically equal to zero, such that:

- $\deg a_{\mathbf{n},j} \leq n_j - 1, j = 1, \ldots, m, \quad \deg a_{\mathbf{n},0} \leq \max(n_j) - 2, \quad (\deg a_{\mathbf{n},j} \leq -1$ means that $a_{\mathbf{n},j} \equiv 0)$
- $a_{\mathbf{n},0}(z) + \sum_{j=1}^m a_{\mathbf{n},j}(z)\widehat{s}_j(z) = \mathcal{O}(1/z^{|\mathbf{n}|}), \quad z \to \infty.$

Analogously, there exist polynomials $Q_{\mathbf{n}}, P_{\mathbf{n},j}, j = 1, \ldots, m$, satisfying:

- $\deg Q_{\mathbf{n}} \leq |\mathbf{n}|, Q_{\mathbf{n}} \not\equiv 0, \quad \deg P_{\mathbf{n},j} \leq |\mathbf{n}| - 1, j = 1, \ldots, m,$
- $Q_{\mathbf{n}}(z)\widehat{s}_j(z) - P_{\mathbf{n},j}(z) = \mathcal{O}(1/z^{n_j+1}), \quad z \to \infty, \quad j = 1, \ldots, m.$

Traditionally, the systems of polynomials $(a_{\mathbf{n},0}, \ldots, a_{\mathbf{n},m})$ and $(Q_{\mathbf{n}}, P_{\mathbf{n},1}, \ldots, P_{\mathbf{n},m})$ have been called type I and type II Hermite-Padé approximants of $(\widehat{s}_1, \ldots, \widehat{s}_m)$, respectively. When $m = 1$ both definitions reduce to that of classical Padé approximation.

From the definition, type II Hermite-Padé approximation is easy to view as an approximating scheme of the vector function $(\widehat{s}_1, \ldots, \widehat{s}_m)$ by considering a sequence of vector rational functions of the form $(P_{\mathbf{n},1}/Q_{\mathbf{n}}, \ldots, P_{\mathbf{n},m}/Q_{\mathbf{n}}), \mathbf{n} \in \Lambda \subset \mathbb{Z}_+^m$, where $Q_{\mathbf{n}}$ is a common denominator for all components. Regarding type I, it is not obvious what is the object to be approximated or even what should be considered as the approximant. Our goal is to clarify these questions providing straightforward analogues of the Markov and Stieltjes theorems in the case when $(s_1, \ldots, s_m)$ is a Nikishin system (see Definition 1.2.2).

When $m = 2$, for multi-indices of the form $\mathbf{n} = (n, n)$ E.M. Nikishin proved in [59] that

$$\lim_{n\to\infty} \frac{P_{\mathbf{n},j}(z)}{Q_{\mathbf{n}}(z)} = \widehat{s}_j(z), \qquad j = 1, 2,$$

uniformly on each compact subset of $\overline{\mathbb{C}} \setminus \Delta_1$. In [16] this result was extended to any Nikishin system of m measures, including generating measures with unbounded support. The convergence for more general sequences of multi-indices was treated in [30, 31] and [42].

In [33, Lemma 2.9] it was shown that if $(\sigma_1, \ldots, \sigma_m)$ is a generator of a Nikishin system then $(\sigma_m, \ldots, \sigma_1)$ is also a generator (as well as any subsystem of consecutive measures drawn from them). When the supports are bounded and consecutive supports do not intersect this is trivially true. In the following, for $1 \leq j \leq k \leq m$ we denote

$$s_{j,k} := \langle \sigma_j, \sigma_{j+1}, \ldots, \sigma_k \rangle, \qquad s_{k,j} := \langle \sigma_k, \sigma_{k-1}, \ldots, \sigma_j \rangle.$$

To state our main results, the natural framework is that of incomplete multi-point type I Hermite-Padé approximation.

Definition 3.1.1. *Let* $(s_{1,1}, \ldots, s_{1,m}) = \mathcal{N}(\sigma_1, \ldots, \sigma_m), \mathbf{n} = (n_1, \ldots, n_m) \in \mathbb{Z}_+^m \setminus \{\mathbf{0}\}$. *Fix* $\ell \in \mathbb{Z}_+$ *and a polynomial* $w_{\mathbf{n}}, \deg w_{\mathbf{n}} \leq |\mathbf{n}| + \max(n_j) - \ell - 2$, *with real coefficients whose zeros lie in* $\mathbb{C} \setminus \Delta_1$. *We say that* $(p_{\mathbf{n},0}, \ldots, p_{\mathbf{n},m})$ *is an incomplete type I multi-point Hermite-Padé approximation of* $(\widehat{s}_{1,1}, \ldots, \widehat{s}_{1,m})$ *with respect to* $w_{\mathbf{n}}$ *if:*

i) $\deg p_{\mathbf{n},j} \leq n_j - 1, j = 1, \ldots, m,\ \deg p_{\mathbf{n},0} \leq n_0 - 1, \quad n_0 := \max_{j=1,\ldots,m}(n_j) - 1$, *not all identically equal to* 0 *(*$n_j = 0$ *implies that* $p_{\mathbf{n},j} \equiv 0$*),*

ii) $\mathcal{A}_{\mathbf{n},0}/w_{\mathbf{n}} \in \mathcal{H}(\mathbb{C} \setminus \Delta_1)$ *and* $\mathcal{A}_{\mathbf{n},0}(z)/w_{\mathbf{n}}(z) = \mathcal{O}(1/z^{|\mathbf{n}|-\ell}), \quad z \to \infty$, *where*

$$\mathcal{A}_{\mathbf{n},j}(z) := p_{\mathbf{n},j}(z) + \sum_{k=j+1}^{m} p_{\mathbf{n},k}(z)\widehat{s}_{j+1,k}(z), \qquad j = 0, \ldots, m-1.$$

When $\ell = 0$ we say that $(p_{\mathbf{n},0}, \ldots, p_{\mathbf{n},m})$ is a type I multi-point Hermite-Padé approximation of $(\widehat{s}_{1,1}, \ldots, \widehat{s}_{1,m})$ with respect to $w_{\mathbf{n}}$ and to distinguish this case we write $(a_{\mathbf{n},0}, \ldots, a_{\mathbf{n},m})$ to denote this vector.

If $\deg w_{\mathbf{n}} = |\mathbf{n}| + \max(n_j) - \ell - 2$ the second part of ii) is automatically fulfilled. Should $\deg w_{\mathbf{n}} = N < |\mathbf{n}| + \max(n_j) - \ell - 2$ then $|\mathbf{n}| + \max(n_j) - \ell - 2 - N$ (asymptotic) interpolation conditions are imposed at ∞. In general $|\mathbf{n}| + \max(n_j) - \ell - 2$ interpolation conditions are imposed at points in $(\mathbb{C} \setminus \Delta_1) \cup \{\infty\}$. The total number of free parameters (the coefficients of the polynomials $p_{\mathbf{n},j}, j = 0. \ldots, m$) is bigger than $|\mathbf{n}| + \max(n_j) - \ell - 2$; therefore, the homogeneous linear system of equations to be solved in order that i)-ii) take place always has a non-trivial solution. Notice that when $w_{\mathbf{n}} \equiv 1$ and $\ell = 0$ we recover the definition given above for classical type I Hermite-Padé approximation.

An analogous definition can be given for type II multi-point Hermite-Padé approximants but we will not dwell into this. Algebraic and analytic properties regarding uniqueness, integral representations, asymptotic behavior, and orthogonality conditions satisfied by type I and type II Hermite-Padé approximants have been studied, for example, in

[16, 26, 27, 28, 30, 31, 32, 33, 42, 59] and [61, Chapter 4], which include the case of multi-point approximation.

We begin with a lemma which allows to give an integral representation for the remainder of type I multi-point Hermite-Padé approximants.

Lemma 3.1.2. *Let* $(s_{1,1},\ldots,s_{1,m}) = \mathcal{N}(\sigma_1,\ldots,\sigma_m)$ *be given. Assume that there exist polynomials with real coefficients* $a_0,\ldots,a_m$ *and a polynomial* w *with real coefficients whose zeros lie in* $\mathbb{C}\setminus\Delta_1$ *such that*

$$\frac{\mathcal{A}(z)}{w(z)} \in \mathcal{H}(\mathbb{C}\setminus\Delta_1) \qquad \textit{and} \qquad \frac{\mathcal{A}(z)}{w(z)} = \mathcal{O}\left(\frac{1}{z^N}\right), \quad z\to\infty,$$

where $\mathcal{A} := a_0 + \sum_{k=1}^m a_k \widehat{s}_{1,k}$ *and* $N \geq 1$. *Let* $\mathcal{A}_1 := a_1 + \sum_{k=2}^m a_k \widehat{s}_{2,k}$. *Then*

$$\frac{\mathcal{A}(z)}{w(z)} = \int \frac{\mathcal{A}_1(x)}{(z-x)}\frac{d\sigma_1(x)}{w(x)}. \tag{3.1}$$

If $N \geq 2$, *we also have*

$$\int x^\nu \mathcal{A}_1(x)\frac{d\sigma_1(x)}{w(x)} = 0, \qquad \nu = 0,\ldots,N-2. \tag{3.2}$$

In particular, $\mathcal{A}_1$ *has at least* $N-1$ *sign changes in* $\overset{\circ}{\Delta}_1$.

Proof. We have

$$\mathcal{A}(z) = a_0(z) + \sum_{k=1}^m a_k(z)\widehat{s}_{1,k}(z) \mp w(z)\int \frac{\mathcal{A}_1(x)}{(z-x)}\frac{d\sigma_1(x)}{w(x)} =$$

$$a_0(z) + \int \frac{\sum_{k=1}^m (w(x)a_k(z) - w(z)a_k(x))ds_{1,k}(x)}{(z-x)w(x)} + w(z)\int \frac{\mathcal{A}_1(x)}{(z-x)}\frac{d\sigma_1(x)}{w(x)}.$$

For each $k = 1,\ldots,m$

$$\left(w(x)a_k(z) - w(z)a_k(x)\right)/(z-x)$$

is a polynomial in z. Therefore,

$$P(z) := a_0(z) + \int \frac{\sum_{k=1}^m (w(x)a_k(z) - w(z)a_k(x))ds_{1,k}(x)}{(z-x)w(x)}$$

represents a polynomial. Consequently

$$\mathcal{A}(z) = P(z) + w(z) \int \frac{\mathcal{A}_1(x) d\sigma_1(x)}{(z-x)w(x)} = w(z)\mathcal{O}(1/z^N), \quad z \to \infty.$$

These equalities imply that

$$P(z) = w(z)\mathcal{O}(1/z), \qquad z \to \infty,$$

Therefore, $\deg P < \deg w$ and is equal to zero at all the zeros of w. Hence $P \equiv 0$. (Should w be a constant polynomial likewise we get that $P \equiv 0$.) Thus, we have proved (3.1).

From our assumptions and (3.1), it follows that

$$\frac{\mathcal{A}(z)}{w(z)} = \int \frac{\mathcal{A}_1(x)}{(z-x)} \frac{d\sigma_1(x)}{w(x)} = \mathcal{O}(1/z^N), \qquad z \to \infty.$$

Suppose that $N \geq 2$. We have the asymptotic expansion

$$\int \frac{\mathcal{A}_1(x)}{(z-x)} \frac{d\sigma_1(x)}{w(x)} =$$

$$\sum_{\nu=0}^{N-2} \frac{d_\nu}{z^{\nu+1}} + \int \frac{x^{N-1}\mathcal{A}_1(x)}{z^{N-1}(z-x)} \frac{d\sigma_1(x)}{w(x)} = \sum_{\nu=0}^{N-2} \frac{d_\nu}{z^{N+1}} + \mathcal{O}(1/z^N), \quad z \to \infty,$$

where

$$d_\nu = \int x^\nu \mathcal{A}_1(x) \frac{d\sigma_1(x)}{w(x)}, \qquad \nu = 0, \ldots, N-2.$$

Therefore,

$$d_\nu = 0, \qquad \nu = 0, \ldots, N-2,$$

which is (3.2).

Suppose that $\mathcal{A}_1$ has at most $\widetilde{N} < N-1$ sign changes in $\overset{\circ}{\Delta}_1$ at the points $x_1, \ldots, x_N$. Take $q(x) = \prod_{k=1}^{\widetilde{N}}(x - x_k)$. According to (3.2)

$$\int q(x)\mathcal{A}_1(x) \frac{d\sigma_1(x)}{w(x)} = 0$$

which is absurd because $q(a_1 + \sum_{k=2}^m a_k \widehat{s}_{2,k})/w$ has constant sign in Δ_1 and σ_1 is a measure with constant sign in Δ_1 whose support contains infinitely many points. Thus, the number of sign changes must be greater or equal to $N-1$ as claimed. $\square$

Using induction, this lemma already allows to prove the AT property for Nikishin system when the multi-indices are in the class

$$\mathbb{Z}_+^m(\circledast) = \{\mathbf{n} \in \mathbb{Z}_+^m : 1 \le j < k \le m \Rightarrow n_k \le n_j + 1\}$$

That result is due to Driver and Stahl (see [27, Theorem 2.4.1]).

Some relations concerning the reciprocal and ratio of Cauchy transforms of measures will be useful. It is known that for each $\sigma \in \mathcal{M}(\Delta)$, where Δ is contained in a half line, there exists a measure $\tau \in \mathcal{M}(\Delta)$ and $\ell(z) = az + b, a = 1/|\sigma|, b \in \mathbb{R}$, such that

$$1/\widehat{\sigma}(z) = \ell(z) + \widehat{\tau}(z), \tag{3.3}$$

where $|\sigma|$ is the total variation of the measure σ. See [46, Appendix] and [74, Theorem 6.3.5] for measures with compact support, and [33, Lemma 2.3] when the support is contained in a half line.

We call τ the inverse measure of σ. Such measures appear frequently in our reasonings, so we will fix a notation to distinguish them. In relation with measures denoted with s they will carry over to them the corresponding sub-indices. The same goes for the polynomials ℓ. For example,

$$1/\widehat{s}_{j,k}(z) = \ell_{j,k}(z) + \widehat{\tau}_{j,k}(z).$$

We also write

$$1/\widehat{\sigma}_\alpha(z) = \ell_\alpha(z) + \widehat{\tau}_\alpha(z).$$

Sometimes we write $\widehat{\langle \sigma_\alpha, \sigma_\beta \rangle}$ in place of $\widehat{s}_{\alpha,\beta}$. In [33, Lemma 2.10], several formulas involving ratios of Cauchy transforms were proved. The most useful ones in this chapter establish that

$$\frac{\widehat{s}_{1,k}}{\widehat{s}_{1,1}} = \frac{|s_{1,k}|}{|s_{1,1}|} - \widehat{\langle \tau_{1,1}, \langle s_{2,k}, \sigma_1 \rangle \rangle}, \qquad 1 = j < k \le m. \tag{3.4}$$

Let $\overset{\circ}{\Delta}$ denote the interior of Δ with the Euclidean topology of the real line. We have

Theorem 3.1.3. *Let* $(s_{1,1}, \ldots, s_{1,m}) = \mathcal{N}(\sigma_1, \ldots, \sigma_m), \mathbf{n} = (n_1, \ldots, n_m) \in \mathbb{Z}_+^m \setminus \{\mathbf{0}\}$, *and* $w_{\mathbf{n}}$, $\deg w_{\mathbf{n}} \le |\mathbf{n}| + \max(n_j) - 2$, *a polynomial with real coefficients whose zeros lie in* $\mathbb{C} \setminus \Delta_1$, *be given. The type I multi-point Hermite-Padé approximation* $(a_{\mathbf{n},0}, \ldots, a_{\mathbf{n},m})$

of $(\widehat{s}_{1,1},\ldots,\widehat{s}_{1,m})$ *with respect to* $w_{\mathbf{n}}$ *is uniquely determined except for a constant factor, and* $\deg a_{\mathbf{n},j}=n_j-1, j=0,\ldots,m$. *Moreover*

$$\int x^{\nu}\mathcal{A}_{\mathbf{n},1}(x)\frac{d\sigma_1(x)}{w_{\mathbf{n}}(x)}=0, \qquad \nu=0,\ldots,|\mathbf{n}|-2, \tag{3.5}$$

which implies that $\mathcal{A}_{\mathbf{n},1}$ *has exactly* $|\mathbf{n}|-1$ *simple zeros in* $\overset{\circ}{\Delta}_1$ *and no other zeros in* $\mathbb{C}\setminus\Delta_2$. *Additionally,*

$$\frac{\mathcal{A}_{\mathbf{n},0}(z)}{w_{\mathbf{n}}(z)}=\int\frac{\mathcal{A}_{\mathbf{n},1}(x)d\sigma_1(x)}{w_{\mathbf{n}}(x)(z-x)} \tag{3.6}$$

and

$$a_{\mathbf{n},0}(z)=-\int\frac{\sum_{j=1}^{m}(w_{\mathbf{n}}(x)a_{\mathbf{n},j}(z)-w_{\mathbf{n}}(z)a_{\mathbf{n},j}(x))ds_{1,j}(x)}{(z-x)w_{\mathbf{n}}(x)}. \tag{3.7}$$

Notice that nothing has been said about the location of the zeros of the polynomials $a_{\mathbf{n},j}$. For special sequences of multi-indices this information can be deduced from the convergence of type I Hermite-Padé approximants.

Proof. Let $(a_{\mathbf{n},0},\ldots,a_{\mathbf{n},m})$ be a type I multi-point Hermite-Padé approximation of the vector of functions $(\widehat{s}_{1,1},\ldots,\widehat{s}_{1,m})$ with respect to $w_{\mathbf{n}}$. From Definition 3.1.1, formulas (3.5) and (3.6) follow directly from (3.2) and (3.1), respectively. Relation (3.7) is obtained from (3.6) solving for $a_{\mathbf{n},0}$.

In the proof of Lemma 3.1.2 we saw that (3.5) implies that $\mathcal{A}_{\mathbf{n},1}$ has at least $|\mathbf{n}|-1$ sign changes in $\overset{\circ}{\Delta}_1$. We have that $(s_{2,2},\ldots,s_{2,m})=\mathcal{N}(\sigma_2,\ldots,\sigma_m)$ forms a Nikishin system. According to [33, Theorem 1.1], $\mathcal{A}_{\mathbf{n},1}$ can have at most $|\mathbf{n}|-1$ zeros in $\mathbb{C}\setminus\Delta_2$. Taking account of what we proved previously, it follows that $\mathcal{A}_{\mathbf{n},1}$ has exactly $|\mathbf{n}|-1$ simple zeros in $\overset{\circ}{\Delta}_1$ and it has no other zero in $\mathbb{C}\setminus\Delta_2$. This is true for any $\mathbf{n}\in\mathbb{Z}_+^m\setminus\{\mathbf{0}\}$.

Suppose that for some $\mathbf{n}\in\mathbb{Z}_+^m\setminus\{\mathbf{0}\}$ and some $j\in\{1,\ldots,m\}$, we have that $\deg a_{\mathbf{n},j}=\widetilde{n}_j-1<n_j-1$. Then, according to [33, Theorem 1.1] $\mathcal{A}_{\mathbf{n},1}$ could have at most $|\mathbf{n}|-n_j+\widetilde{n}_j-1\leq|\mathbf{n}|-2$ zeros in $\mathbb{C}\setminus\Delta_2$. This is absurd because we have proved that it has $|\mathbf{n}|-1$ zeros in $\overset{\circ}{\Delta}_1$.

Now, suppose that for some $\mathbf{n}\in\mathbb{Z}_+^m\setminus\{\mathbf{0}\}$, there exist two non collinear type I multi-point Padé approximants $(a_{\mathbf{n},0},\ldots,a_{\mathbf{n},m})$ and $(\widetilde{a}_{\mathbf{n},0},\ldots,\widetilde{a}_{\mathbf{n},m})$ of $(\widehat{s}_{1,1},\ldots,\widehat{s}_{1,m})$ with respect to $w_{\mathbf{n}}$. From (3.7) it follows that $(a_{\mathbf{n},1},\ldots,a_{\mathbf{n},m})$ and $(\widetilde{a}_{\mathbf{n},1},\ldots,\widetilde{a}_{\mathbf{n},m})$ are not collinear. We know that $\deg a_{\mathbf{n},j}=\deg\widetilde{a}_{\mathbf{n},j}=n_j-1, j=1,\ldots,m$. Consequently, there exist some constant C such that $(a_{\mathbf{n},1} - C\widetilde{a}_{\mathbf{n},1}, \ldots, a_{\mathbf{n},m} - C\widetilde{a}_{\mathbf{n},m}) \neq \mathbf{0}$ and $\deg(a_{\mathbf{n},j} - C\widetilde{a}_{\mathbf{n},j}) < n_j - 1$ for some $j \in \{1, \ldots, m\}$. By linearity, $(a_{\mathbf{n},0} - C\widetilde{a}_{\mathbf{n},0}, \ldots, a_{\mathbf{n},m} - C\widetilde{a}_{\mathbf{n},m})$ is a multi-point type I Hermite-Padé approximant of $(\widehat{s}_{1,1}, \ldots, \widehat{s}_{1,m})$ with respect to $w_{\mathbf{n}}$. This is not possible because $\deg(a_{\mathbf{n},j} - C\widetilde{a}_{\mathbf{n},j}) < n_j - 1$. Therefore, non-collinear solutions cannot exist.

We still need to show that $\deg a_{\mathbf{n}_0} = n_0 - 1$. To this end we need to transform $\mathcal{A}_{\mathbf{n},0}$. Let j be the first component of $\mathbf{n}$ such that $n_j = \max_{k=1,\ldots,m} n_k$. Since $n_0 = n_j - 1$, we have that either $j = 1$ or $n_0 \geq n_k, k = 1, \ldots, j-1$. If $j = 1$, using (3.3) and (3.4) it follows that

$$\mathcal{B}_{\mathbf{n},0} := \frac{\mathcal{A}_{\mathbf{n},0}}{\widehat{s}_{1,1}} = \ell_{1,1} a_{\mathbf{n},0} + \sum_{k=1}^{m} \frac{|s_{1,k}|}{|s_{1,1}|} a_{\mathbf{n},k} + a_{\mathbf{n},0}\widehat{\tau}_{1,1} - \sum_{k=2}^{m} a_{\mathbf{n},k} \widehat{\langle \tau_{1,1}, \langle s_{2,k}, \sigma_1 \rangle \rangle},$$

where

$$\mathcal{B}_{\mathbf{n},0}/w_{\mathbf{n}} \in \mathcal{H}(\mathbb{C} \setminus \Delta_1), \qquad \mathcal{B}_{\mathbf{n},0}(z)/w_{\mathbf{n}}(z) = \mathcal{O}(1/z^{|\mathbf{n}|-1}), \quad z \to \infty.$$

Using Lemma 3.1.2 it follows that

$$\int x^{\nu} \mathcal{B}_{\mathbf{n},1}(x) \frac{d\tau_{1,1}(x)}{w_{\mathbf{n}}(x)}, \qquad \nu = 0, \ldots, |\mathbf{n}| - 3.$$

where $\mathcal{B}_{\mathbf{n},1} = a_{\mathbf{n},0} - \sum_{k=2}^{m} a_{\mathbf{n},k} \widehat{\langle \langle \sigma_2, \sigma_1 \rangle, \sigma_3, \ldots, \sigma_k \rangle}$. Hence $\mathcal{B}_{\mathbf{n},1}$ has at least $|\mathbf{n}| - 2$ sign changes in $\overset{\circ}{\Delta}_1$. According to [33, Theorem 1.1] the linear form $\mathcal{B}_{\mathbf{n},1}$ has at most $\deg a_{\mathbf{n},0} + n_2 + \cdots + n_m$ zeros in all of $\mathbb{C} \setminus \Delta_2$. Should $\deg a_{\mathbf{n},0} \leq n_0 - 2$, we would have that $\deg a_{\mathbf{n},0} + n_2 + \cdots + n_m \leq |\mathbf{n}| - 3$ which contradicts that $\mathcal{B}_{\mathbf{n},1}$ has at least $|\mathbf{n}| - 2$ zeros in $\overset{\circ}{\Delta}_1$. Thus, when $j = 1$ it is true that $\deg a_{\mathbf{n},0} = n_0 - 1$. For $j > 1$, the proof is similar.

Suppose that j, as defined in the previous paragraph, is ≥ 2. Then, either $n_0 = n_k, k = 1, \ldots, j-1$ or there exists $\overline{\jmath} < j$ for which $n_0 = n_k, k = 1, \ldots, \overline{\jmath} - 1$ and $n_0 > n_{\overline{\jmath}}$. In the first case, applying [33, Lemma 2.12], we obtain that there exists a Nikishin system $(s_{1,1}^*, \ldots, s_{1,m}^*) = \mathcal{N}(\sigma_1^*, \ldots, \sigma_m^*)$, a multi-index $\mathbf{n}^* = (n_0^*, \ldots, n_m^*) \in \mathbb{Z}_+^{m+1}$ which is a permutation of $\mathbf{n}$ with $n_0^* = n_j$, and polynomials with real coefficients $a_{\mathbf{n},k}^*$, $\deg a_{\mathbf{n},k}^* \leq n_k^* - 1, k = 0, \ldots, m$, such that

$$\frac{\mathcal{A}_{\mathbf{n},0}}{\widehat{s}_{1,j}} = a_{\mathbf{n},0}^* + \sum_{k=1}^{m} a_{\mathbf{n},k}^* \widehat{s}_{1,k}^*$$

Due to the structure of the values of the components of the multi-index we have that $a^*_{\mathbf{n},j} = (-1)^j a_{\mathbf{n},0}$ and $n^*_j = n_0$ (see also formula (31) in [32]). We can proceed as before and find that $\deg a^*_{\mathbf{n},j} = n^*_j - 1, j = 1,\ldots,m$. In particular, $\deg a_{\mathbf{n},0} = n_0 - 1$. In the other case, [33, Lemma 2.12] gives that

$$\frac{\mathcal{A}_{\mathbf{n},0}}{\widehat{s}_{1,j}} = a^*_{\mathbf{n},0} + \sum_{k=1}^{m} a^*_{\mathbf{n},k}\widehat{s}^*_{1,k}$$

where $a^*_{\mathbf{n},\bar{\jmath}} = \pm a_{\mathbf{n},0} + C a_{\mathbf{n},\bar{\jmath}}$, $C \neq 0$ is some constant, and $n^*_{\bar{\jmath}} = n_0$, Repeating the arguments employed above, we obtain that $\deg a^*_{\mathbf{n},j} = n^*_j - 1, j = 1,\ldots,m$. In particular, $\deg a^*_{\mathbf{n},\bar{\jmath}} = n^*_{\bar{\jmath}} - 1 = n_0 - 1$ wich implies that $\deg a_{\mathbf{n},0} = n_0 - 1$ since $\deg a^*_{\mathbf{n},\bar{\jmath}} = \deg a_{\mathbf{n},0}$. $\square$

Remark 3.1.4. *We wish to point out that in the statement of [33, Theorem 1.1] there is a missprint on the last line where $\mathbb{C}$ should replace $\overline{\mathbb{C}}$. That is, it should refer to zeros at finite points. This can be checked looking at the statements of [33, Lemmas 2.1, 2.2] and the proof of [33, Theorem 1.1] itself.*

The following result has independent interest and will be used in combination with Lemma 3.2.2 in the proof of Theorem 3.2.1. Recall that a positive measure s, $\mathrm{supp}(s) \subset \mathbb{R}_+$ is said to satisfy Carleman's condition if

$$\sum_{n>1} c_n^{-1/2n} = \infty,$$

where $c_n = \int x^n ds(x)$ is the n-th moment. A negative measure s is said to satisfy Carleman's condition if $-s$ satisfies that property. A measure s whose support is contained in a half line satisfies Carleman's condition if after an affine transformation which takes the convex hull of s to a subset of $\mathbb{R}_+$ the image measure satifies Carleman's condition. It is easy to verify that this definiton does not depend on the affine transformation taken.

Theorem 3.1.5. *Let $\sigma_1, \sigma_2 \in \mathcal{M}(\Delta)$, where Δ_1 is contained in a half line. If σ_1 satisfies Carleman's condition so do $\langle \sigma_1, \sigma_2 \rangle$ and τ_1, where τ_1 is the inverse measure of σ_1.*

Proof. Without loss of generality, we can assume that $\Delta \subset \mathbb{R}_+$ and that σ_1 is positive. Let $(c_n)_{n\in\mathbb{Z}_+}$ and $(\widetilde{c}_n)_{n\in\mathbb{Z}_+}$ denote the sequences of moments of σ_1 and $s_{1,2} = \langle \sigma_1, \sigma_2 \rangle$, respectively. Since $\widehat{\sigma}_2$ has constant sign on $\mathbb{R}_+$, we have that

$$|\widetilde{c}_n| = \int x^n |\widehat{\sigma}_2(x)| d\sigma_1(x) \leq \int_0^1 x^n |\widehat{\sigma}_2(x)| d\sigma_1(x) + \int_1^\infty x^n |\widehat{\sigma}_2(x)| d\sigma_1(x) \leq |s_{1,2}| + C c_n,$$

where $C = \max\{|\widehat{\sigma}_2(x)| : x \in [1,+\infty)\} < \infty$ because $\lim_{x\to\infty} \widehat{\sigma}_2(x) = 0$. Consequently,

$$\sum_{n\geq 1} |\widetilde{c}_n|^{-1/2n} \geq \sum_{n\geq 1} (|s_{1,2}| + Cc_n)^{-1/2n} \tag{3.8}$$

$$\geq \sum_{\{n:Cc_n<|s_{1,2}|\}} (2|s_{1,2}|)^{-1/2n} + \sum_{\{n:Cc_n\geq|s_{1,2}|\}} (2Cc_n)^{-1/2n}. \tag{3.9}$$

If the first sum after the last inequality contains infinitely many terms then that sum is already divergent. If it has finitely many terms then Carleman's condition for σ_1 guarantees that the second sum is divergent. Thus, $s_{1,2}$ satisfies Carleman's condition.

To prove the second part we need to express the moments $(d_n)_{n\in\mathbb{Z}_+}$ of τ_1 in terms of the moments of σ_1. In the proof of [33, Lemma 2.3] it was proved that the moments $(d_n)_{n\in\mathbb{Z}_+}$ are finite (since all the moments of σ_1 are finite) and they can be obtained solving recursively the system of equations

$$\begin{array}{rcl} 1 & = & d_{-2}c_0 \\ 0 & = & d_{-2}c_1 + d_{-1}c_0 \\ 0 & = & d_{-2}c_2 + d_{-1}c_1 + d_0c_0 \\ \vdots & = & \vdots \\ 0 & = & d_{-2}c_{n+2} + d_{-1}c_{n+1} + \cdots + d_nc_0 \ . \end{array} \tag{3.10}$$

(The values of d_{-2} and d_{-1} turn out to be the coefficients a and b, respectively, of the polynomial ℓ_1 in the decomposition (4.17) of $1/\widehat{\sigma}_1$.) Read the paragraph after formula (9) in [33].

To find d_n we apply Cramer's rule and we get

$$d_n = (-1)^n\Omega_n/c_0^{n+3} \tag{3.11}$$

where c_0^{n+3} gives the value of the determinant of the system and

$$\Omega_n = \begin{vmatrix} c_1 & c_0 & 0 & \cdots \\ c_2 & c_1 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \\ c_{n+2} & c_{n+1} & \cdots & c_1 \end{vmatrix}$$

is the determinant of a lower Hessenberg matrix of dimension $n+2$ with constant diagonal terms. The expansion of the determinant Ω_n has several characteristics:

- It has exactly 2^{n+1} non zero terms.
- For each $n \geq 0$, the sum of the subindices of each non zero term equals $n+2$ (if a factor is repeated its subindex is counted as many times as it is repeated).
- The number of factors in each term is equal to $n+2$.

The last assertion is trivial. To calculate the number of non zero terms notice that from the first row we can only choose 2 non zeros entries. Once this is done, from the second row we can only choose 2 non zero entries, and so forth, until we get to the last row where we only have left one non zero entry to choose.

Regarding the second assertion we use induction. When $n = 0$ it is obvious. Assume that each non zero term in the expansion of Ω_n has the property that the sum of its subindices equals $n+2$ and let us show that each non zero term in the expansion of Ω_{n+1} has the property that the sum of its subindices equals $n+3$. Expanding Ω_{n+1} by its first row we obtain

$$\Omega_{n+1} = c_1\Omega_n - c_0\Omega_n^*,$$

where Ω_n^* is obtained substituting the first column of Ω_n by the column vector

$$(c_2, \ldots, c_{n+3})^T$$

(the superscript T means taking transpose). Using the induction hypothesis it easily follows that for each term arising from $c_1\Omega_n$ and $c_0\Omega_n^*$ the sum of its subindices must equal $n+3$.

Using the properties proved above we obtain that the general expression of Ω_n is

$$\Omega_n = \sum_{j=1}^{n+2} \sum_{\alpha_1+\cdots+\alpha_j=n+2} \varepsilon_\alpha c_0^{n+2-j} c_{\alpha_1}\cdots c_{\alpha_j},$$

where $\alpha = (\alpha_1, \ldots, \alpha_j), 1 \leq \alpha_k \leq n+2, 1 \leq k \leq j$ and $\varepsilon_\alpha = \pm 1$. Thus

$$|\Omega_n| \leq \sum_{j=1}^{n+2} \sum_{\alpha_1+\cdots+\alpha_j=n+2} c_0^{n+2-j} c_{\alpha_1}\cdots c_{\alpha_j}. \tag{3.12}$$

From all these terms there is only one which contains the factor c_{n+2} and that is when $j = 1$. That term is $c_0^{n+1}c_{n+2}$. In the rest of the terms $1 \leq \alpha_k \leq n+1$. Let us prove that

$$c_0^{n+2-j} c_{\alpha_1}\cdots c_{\alpha_j} \leq c_0^{n+1}c_{n+2} \qquad \text{for all} \qquad \alpha. \tag{3.13}$$

In fact, using the Holder inequality on each factor except the first, it follows that

$$c_0^{n+2-j} c_{\alpha_1} \cdots c_{\alpha_j} \leq c_0^{n+2-j} \left(\int x^{n+2} d\sigma_1(x) \right)^{\sum_{k=1}^{j} \alpha_k/(n+2)} \left(\int d\sigma_1(x) \right)^{j-(\sum_{k=1}^{j} \alpha_k)/(n+2)}.$$

It rests to employ that $\sum_{k=1}^{j} \alpha_k = n+2$ to complete the proof of (3.13).

From (3.11), (3.12), and (3.13), we have that

$$d_n \leq 2^{n+1} c_{n+2}/c_0^2$$

and the Carleman condition for τ_1 readily follows. □

3.2 A Markov theorem for type I Hermite-Padé approximants

We are ready to state the main results of this chapter.

Theorem 3.2.1. *Let* $S = (s_{1,1}, \ldots, s_{1,m}) = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$, $\Lambda \subset \mathbb{Z}_+^m$ *an infinite sequence of distinct muti-indices, and* $(w_{\mathbf{n}})_{\mathbf{n}\in\Lambda}, \deg w_{\mathbf{n}} \leq |\mathbf{n}| + \max(n_j) - 2$, *a sequence of polynomials with real coefficients whose zeros lie in* $\mathbb{C} \setminus \Delta_1$, *be given. Consider the corresponding sequence* $(a_{\mathbf{n},0}, \ldots, a_{\mathbf{n},m})$, $\mathbf{n} \in \Lambda$, *of type I multi-point Hermite-Padé approximants of* S *with respect to* $(w_{\mathbf{n}})_{\mathbf{n}\in\Lambda}$. *Assume that*

$$\sup_{\mathbf{n}\in\Lambda} \left(\max_{j=1,\ldots,m} (n_j) - \min_{k=1,\ldots,m} (n_k) \right) \leq C < \infty, \tag{3.14}$$

and that either Δ_{m-1} *is bounded away from* Δ_m *or* σ_m *satisfies Carleman's condition. Then,*

$$\lim_{\mathbf{n}\in\Lambda} \frac{a_{\mathbf{n},j}}{a_{\mathbf{n},m}} = (-1)^{m-j} \widehat{s}_{m,j+1}, \qquad j = 0, \ldots, m-1, \tag{3.15}$$

uniformly on each compact subset $K \subset \mathbb{C} \setminus \Delta_m$. *The accumulation points of sequences of zeros of the polynomials* $a_{\mathbf{n},j}, j = 0, \ldots, m, \mathbf{n} \in \Lambda$ *are contained in* $\Delta_m \cup \{\infty\}$. *Additionally,*

$$\lim_{\mathbf{n}\in\Lambda} \frac{\mathcal{A}_{\mathbf{n},j}}{a_{\mathbf{n},m}} = 0, \qquad j = 0, \ldots, m-1, \tag{3.16}$$

uniformly on each compact subset $K \subset \mathbb{C} \setminus (\Delta_{j+1} \cup \Delta_m)$.

That is, for $j = 0, \ldots, m-1$ the sequences of rational functions $(a_{\mathbf{n},j}/a_{\mathbf{n},m})$, $\mathbf{n} \in \Lambda$ allow to recover the Cauchy transforms of the measures in $\mathcal{N}(\sigma_m, \ldots, \sigma_1)$ in contrast with the sequences $(P_{\mathbf{n},j}/Q_{\mathbf{n}})$, $\mathbf{n} \in \Lambda, j = 1, \ldots, m$, of type II multi-point Hermite-Padé approximants which recover the Cauchy transforms of the measures in $\mathcal{N}(\sigma_1, \ldots, \sigma_m)$.

Rahkmanov and Suetin, in [63] and [64] obtain a related result. The first one of these papers announces the results contained in the second one. Those papers deal with the study of type I Hermite-Padé approximants for an interesting class of systems of two functions $(m = 2)$ which form a generalized Nikishin system in the sense that the second generating measure lives on a symmetric (with respect to the real line) compact set which does not separate the complex plane and is made up of finitely many analytic arcs. The authors obtain the logarithmic asymptotic of the sequences of Hermite-Padé polynomials $a_{\mathbf{n},j}, j = 1, 2,$ and an analogue of (3.15) for $j = 1$. Convergence is proved in capacity (see [63, Theorem 1] and [64, Theorem 1]. We wish to underline that in Theorem 3.2.1 no special analytic properties is required from the generating measures of the Nikishin system.

The notion of convergence in Hausdorff content plays a central role in the rest of this thesis. Let B be a subset of the complex plane $\mathbb{C}$. By $\mathcal{U}(B)$ we denote the class of all coverings of B by at most a numerable set of disks. Set

$$h(B) = \inf \left\{ \sum_{i=1}^{\infty} |U_i| \, : \, \{U_i\} \in \mathcal{U}(B) \right\},$$

where $|U_i|$ stands for the radius of the disk U_i. The quantity $h(B)$ is called the 1-dimensional Hausdorff content (or simply Hausdorff content) of the set B.

From the definition it is easily seen that the 1-dimensional Hausdorff content of the disk of radius R is exactly R.

Let $P(z) = \prod_{k=1}^{l}(z - a_k)$ be a monic polynomial of degree l. Consider the set $E = \{z : |P(z)| \leq R\}$. It is easy to check that $E \subset \cup_{k=1}^{l}\{z : |z - a_k| \leq R^{1/l}\}$. Since Hausdorff content is a subadditive function of sets, it readly follows that $h(E) \leq lR^{1/l}$.

The 1-dimensional Hausdorff content is monotonic and subadditive, however, it is not a measure because, in general, the σ-additive property does not hold.

Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence of complex functions defined on a domain $D \subset \mathbb{C}$ and φ another function defined on D (the value ∞ is permitted). We say that $(\varphi_n)_{n\in\mathbb{N}}$ converges in Hausdorff content to the function φ inside D if for each compact subset K of D and for each $\varepsilon > 0$, we have

$$\lim_{n\to\infty} h\{z \in K : |\varphi_n(z) - \varphi(z)| > \varepsilon\} = 0$$

(by convention $\infty \pm \infty = \infty$). We denote this writing $h\text{-}\lim_{n\to\infty} \varphi_n = \varphi$ inside D.

Another common way, in the complex plane, to measure a the size of set $E \subset \mathbb{C}$ is through the concept of logaritmic capacity. That definition can be introduced in several manners, we follow the notation and the definitions used in Chapter 5, [69].

Let K be a compact set of the complex plane $\mathbb{C}$. Let $\mathcal{M}(K)$ denote the space of all finite Borel meausure whose support is contained in K ($\text{supp}(\mu) \subset K$). We will denote by $|\mu| = \mu(K)$, the total variation of the measure μ. It is straightfoward that

$$I(\mu) = \int\int \log \frac{1}{|z-\zeta|} d\mu(z) d\mu(\zeta) \geq |\mu|^2 \log \frac{1}{d} \tag{3.17}$$

where $d = \max\{|z - x| : z, x \in K\}$. We call $I(\mu)$ the energy of the meausure μ. From (3.17) we deduce that

$$I(K) = \inf_{|\mu|=1} \{I(\mu) : \mu \in \mathcal{M}(K)\} \geq \log \frac{1}{d} > -\infty \tag{3.18}$$

We call logaritmic capacity of the compact set K the number

$$cap(K) = \exp^{-I(K)}.$$

If U denotes an open set, its capacity is defined as

$$cap(U) = \sup\{cap(K) : K \subset U \text{ and } K \text{ is compact}\}.$$

For, an arbitrary set F

$$cap(F) = \inf\{cap(U) : U \supset F \text{ and } F \text{ is open}\}$$

From the definition it readly follows that the capacity of sets is a monotonic function. There are sets of logaritmic capacity equal to zero, for example, if E is a set with only a numerable number of points, then every probabiltity measure μ supported on E has at least one mass point; otherwise by the $\sigma - additivity$ we have that $|\mu| = 0$. It is easy to check that the energy of a measure with at least one mass point is equal to infinity which implies that every numerable set has zero logaritmic capacity. There are non numerable sets with zero logaritmic capacity, but from the definition it follows that the unique measure with finite energy supported on a set with zero logaritmic capacity is the zero measure. There are sets of Lebesgue measure zero that don not have zero logaritmic capacity; however, the sets with zero logaritmic capacity have zero Lebesgue measure.

Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence of complex functions defined on a domain $D \subset \mathbb{C}$ and φ another function defined on D (the value ∞ is permitted). We say that $(\varphi_n)_{n\in\mathbb{N}}$ converges in logaritmic capacity to the function φ inside D if for each compact subset K of D and for each $\varepsilon > 0$, we have

$$\lim_{n\to\infty} cap\{z \in K : |\varphi_n(z) - \varphi(z)| > \varepsilon\} = 0$$

We denote this writing $\mathcal{C}\text{-}\lim_{n\to\infty} \varphi_n = \varphi$ inside D.

Recall that if P is a monic polynomial of degree l and $E = \{z : |P(z)| \leq R\}$ then $h(E) \leq lR^{1/l}$. For the capacity it is well known that $cap(E) = R^{1/l}$. These relations and the mononicity of the capacity and 1-Hausdorff content are the basic used to prove convergence in either sense of rational approximants. This allows us to assert that in the rest of the thesis all results stated in terms of convergence in 1-Hausdorff content are equally true with convergence in capacity. We work with 1-Hausdorff content for convenience since it is a much more geometric object.

To obtain Theorem 3.2.1 we first prove (3.15) with convergence in Hausdorff content in place of uniform convergence (see Lemma 3.2.3 below). We need the following notion.

Let $s \in \mathcal{M}(\Delta)$ where Δ is contained in a half line of the real axis. Fix an arbitrary $\kappa \geq -1$. Consider a sequence of polynomials $(w_n)_{n\in\Lambda}, \Lambda \subset \mathbb{Z}_+$, such that $\deg w_n = \kappa_n \leq 2n + \kappa + 1$, whose zeros lie in $\mathbb{R} \setminus \Delta$. Let $(R_n)_{n\in\Lambda}$ be a sequence of rational functions $R_n = p_n/q_n$ with real coefficients satisfying the following conditions for each $n \in \Lambda$:

a) $\deg p_n \leq n + \kappa, \quad \deg q_n \leq n, \quad q_n \not\equiv 0,$

b) $(q_n\widehat{s} - p_n)(z)/w_n = \mathcal{O}\left(1/z^{n+1-\ell}\right) \in \mathcal{H}(\mathbb{C} \setminus \Delta), z \to \infty,$ where $\ell \in \mathbb{Z}_+$ is fixed.

We say that $(R_n)_{n\in\Lambda}$ is a sequence of incomplete diagonal multi-point Padé approximants of $\widehat{s}$.

Notice that in this construction for each $n \in \Lambda$ the number of free parameters equals $2n+\kappa+2$ whereas the number of homogeneous linear equations to be solved in order to find q_n and p_n is equal to $2n + \kappa - \ell + 1$. When $\ell = 0$ there is only one more parameter than equations and R_n is defined uniquely coinciding with a (near) diagonal multi-point Padé approximation. When $\ell \geq 1$ uniqueness is not guaranteed, thus the term incomplete.

For sequences of incomplete diagonal multi-point Padé approximants, the following Stieltjes type theorem was proved in [16, Lemma 2] in terms of convergence in logaritmic capacity and we reformulate it using 1-Hausdorff content.

Lemma 3.2.2. *Let $s \in \mathcal{M}(\Delta)$ be given where Δ is contained in a half line. Assume that $(R_n)_{n\in\Lambda}$ satisfies* a)-b) *and either the number of zeros of w_n lying on a bounded segment of $\mathbb{R} \setminus \Delta$ tends to infinity as $n \to \infty, n \in \Lambda$, or s satisfies Carleman's condition. Then*

$$h - \lim_{n\in\Lambda} R_n = \widehat{s}, \qquad \textit{inside} \qquad \mathbb{C} \setminus \Delta.$$

Our first step consists in proving a weaker version of (3.15).

Lemma 3.2.3. *Let $\mathrm{s} = (s_{1,1}, \ldots, s_{1,m}) = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$ and $\Lambda \subset \mathbb{Z}_+^m$ be an infinite sequence of distinct multi-indices. Fix $\ell \in \mathbb{Z}_+$ and a sequence $(w_{\mathbf{n}})_{\mathbf{n}\in\Lambda}, \deg w_{\mathbf{n}} \leq |\mathbf{n}| - \ell - 1$, of polynomials whose zeros lie in $\mathbb{R} \setminus \Delta_1$. Consider a sequence of incomplete type I multi-point Hermite-Padé approximants of* s *with respect to $(w_{\mathbf{n}})_{\mathbf{n}\in\Lambda}$. Assume that* (3.14) *takes place and that either Δ_{m-1} is bounded away from Δ_m or σ_m satisfies Carleman's condition. Then, for each fixed $j = 0, \ldots, m-1$*

$$h - \lim_{\mathbf{n}\in\Lambda} \frac{p_{\mathbf{n},j}}{p_{\mathbf{n},m}} = (-1)^{m-j} \widehat{s}_{m,j+1}, \quad h - \lim_{n\in\Lambda} \frac{p_{\mathbf{n},m}}{p_{\mathbf{n},j}} = \frac{(-1)^{m-j}}{\widehat{s}_{m,j+1}}, \tag{3.19}$$

inside $\mathbb{C} \setminus \Delta_m$. There exists a constant C_1, independent of Λ, such that for all $\mathbf{n} \in \Lambda$, the polynomials $p_{\mathbf{n},j}, j = 0, \ldots, m$, have at least $(|\mathbf{n}|/(m+1)) - C_1$ zeros in $\mathring{\Delta}_m$.

Proof. If $m = 1$ the statement reduces directly to Lemma 3.2.2, so without loss of generality we can assume that $m \geq 2$. Fix $\mathbf{n} \in \Lambda$.

From Lemma 3.1.2 it follows that $\mathcal{A}_{\mathbf{n},1}$ has at least $|\mathbf{n}| - D - 1$ simple zeros in the interior of Δ_1. Therefore, there exists a polynomial $w_{\mathbf{n},1}, \deg w_{\mathbf{n},1} = |\mathbf{n}| - D - 1$, whose zeros lie on Δ_1 such that

$$\frac{\mathcal{A}_{\mathbf{n},1}}{w_{\mathbf{n},1}} \in \mathcal{H}(\mathbb{C} \setminus \Delta_2). \tag{3.20}$$

Set $\overline{n}_j = \max\{n_k : k = j, \ldots, m\}$. Taking into account the degrees of the polynomials $p_{\mathbf{n},j}$ and $w_{\mathbf{n},1}$ it follows that

$$\frac{\mathcal{A}_{\mathbf{n},1}}{w_{\mathbf{n},1}} = \mathcal{O}\left(\frac{1}{z^{|\mathbf{n}|-D-\overline{n}_1}}\right), \qquad z \to \infty. \tag{3.21}$$

From (3.20), (3.21), and Lemma 3.1.2 we have that $\mathcal{A}_{\mathbf{n},2}$ has at least $|\mathbf{n}|-D-\overline{n}_1-1$ sign changes in $\overset{\circ}{\Delta}_2$. Therefore, there exists a polynomial $w_{\mathbf{n},2}$, $\deg w_{\mathbf{n},2}=|\mathbf{n}|-D-\overline{n}_1-1$, whose zeros lie on Δ_2, such that

$$\frac{\mathcal{A}_{\mathbf{n},2}}{w_{\mathbf{n},2}} \in \mathcal{H}(\mathbb{C}\setminus\Delta_3), \quad \text{and} \quad \frac{\mathcal{A}_{\mathbf{n},2}}{w_{\mathbf{n},2}} = \mathcal{O}\left(\frac{1}{z^{|\mathbf{n}|-D-\overline{n}_1-\overline{n}_2}}\right), \quad z\to\infty.$$

Iterating this process, using Lemma 3.1.2 several times, on step j, $j\in\{1,\ldots,m\}$, we find that there exists a polynomial $w_{\mathbf{n},j}$, $\deg w_{\mathbf{n},j}=|\mathbf{n}|-D-\overline{n}_1-\cdots-\overline{n}_{j-1}-1$, whose zeros are points where $\mathcal{A}_{\mathbf{n},j}$ changes sign on Δ_j such that

$$\frac{\mathcal{A}_{\mathbf{n},j}}{w_{\mathbf{n},j}} \in \mathcal{H}(\mathbb{C}\setminus\Delta_{j+1}), \quad \text{and} \quad \frac{\mathcal{A}_{\mathbf{n},j}}{w_{\mathbf{n},j}} = \mathcal{O}\left(\frac{1}{z^{|\mathbf{n}|-D-\overline{n}_1-\cdots-\overline{n}_j}}\right), \quad z\to\infty. \tag{3.22}$$

This process concludes as soon as $|\mathbf{n}|-D-\overline{n}_1-\cdots-\overline{n}_j\le 0$. Since $\lim_{\mathbf{n}\in\Lambda}|\mathbf{n}|=\infty$, because of (3.14) we can always take m steps for all $\mathbf{n}\in\Lambda$ with $|\mathbf{n}|$ sufficiently large. In what follows, we only consider such $\mathbf{n}$'s.

When $n_1=\overline{n}_1\ge\cdots\ge n_m=\overline{n}_m$, we obtain that $\mathcal{A}_{\mathbf{n},m}\equiv p_{\mathbf{n},m}$ has at least n_m-D-1 sign changes on Δ_m. If $D=0$ since $\deg p_{\mathbf{n},m}\le n_m-1$ this means that $\deg p_{\mathbf{n},m}=n_m-1$ and all its zeros lie on Δ_m. (In fact, in this case we can prove that $\mathcal{A}_{\mathbf{n},j}$, $j=1,\ldots,m$ has exactly $|\mathbf{n}|-n_1-\cdots-n_{j-1}$ zeros in $\mathbb{C}\setminus\Delta_{j+1}$ that they are all simple and lie in the interior of Δ_j, where $\Delta_{m+1}=\emptyset$).

In general, $p_{\mathbf{n},m}$ has at least $|\mathbf{n}|-D-\overline{n}_1-\cdots-\overline{n}_{m-1}-1$ sign changes on Δ_m; therefore, the number of zeros of $p_{\mathbf{n},m}$ which may lie outside of Δ_m is bounded by

$$\deg p_{\mathbf{n},m}-(|\mathbf{n}|-D-\overline{n}_1-\cdots-\overline{n}_{m-1}-1)\le\sum_{k=1}^{m-1}\overline{n}_k-n_k\le(m-1)C+D,$$

where C is the constant given in (3.14), which does not depend on $\mathbf{n}\in\Lambda$.

For $j=m-1$ there exists $w_{\mathbf{n},m-1}$, $\deg w_{\mathbf{n},m-1}=|\mathbf{n}|-D-\overline{n}_1-\cdots-\overline{n}_{m-2}-1$, whose zeros lie on Δ_{m-1} such that

$$\frac{\mathcal{A}_{\mathbf{n},m-1}}{w_{\mathbf{n},m-1}}=\frac{p_{\mathbf{n},m-1}+p_{\mathbf{n},m}\widehat{\sigma}_m}{w_{\mathbf{n},m-1}}\in\mathcal{H}(\mathbb{C}\setminus\Delta_m),$$

and

$$\frac{\mathcal{A}_{\mathbf{n},m-1}}{w_{\mathbf{n},m-1}}=\mathcal{O}\left(\frac{1}{z^{|\mathbf{n}|-D-\overline{n}_1-\cdots-\overline{n}_{m-1}}}\right),\ z\to\infty,$$

where $\deg p_{\mathbf{n},m-1} \leq n_{m-1} - 1, \deg p_{\mathbf{n},m} \leq n_m - 1$. Thus, using (3.14) it is easy to check that $(p_{\mathbf{n},m-1}/p_{\mathbf{n},m})_{n\in\Lambda}$ forms a sequence of incomplete diagonal multi-point Padé approximants of $-\widehat{\sigma}_m$ satisfying a)-b) with appropriate values of n, κ and ℓ. Due to Theorem 3.1.5 and Lemma 3.2.2 it follows that

$$h - \lim_{n\in\Lambda} \frac{p_{\mathbf{n},m-1}}{p_{\mathbf{n},m}} = -\widehat{\sigma}_m, \qquad \text{inside} \qquad \mathbb{C} \setminus \Delta_m.$$

Dividing by $\widehat{\sigma}_m$ and using (4.17), we also have

$$\frac{\mathcal{A}_{\mathbf{n},m-1}}{\widehat{\sigma}_m w_{\mathbf{n},m-1}} = \frac{p_{\mathbf{n},m-1}\widehat{\tau}_m + b_{\mathbf{n},m-1}}{w_{\mathbf{n},m-1}} \in \mathcal{H}(\mathbb{C} \setminus \Delta_m),$$

where $b_{\mathbf{n},m-1} = p_{\mathbf{n},m} + \ell_m p_{\mathbf{n},m-1}$ and

$$\frac{\mathcal{A}_{\mathbf{n},m-1}}{\widehat{\sigma}_m w_{\mathbf{n},m-1}} = \mathcal{O}\left(\frac{1}{z^{|\mathbf{n}|-D-\overline{n}_1-\cdots-\overline{n}_{m-1}-1}}\right), \; z \to \infty.$$

Consequently, $(b_{\mathbf{n},m-1}/p_{\mathbf{n},m-1})_{n\in\Lambda}$ forms a sequence of incomplete diagonal multi-point Padé approximants of $-\widehat{\tau}_m$ satisfying a)-b) with appropriate values of n, κ and ℓ. Then Theorem 3.1.5 and Lemma 3.2.2 imply that

$$h - \lim_{n\in\Lambda} \frac{b_{\mathbf{n},m-1}}{p_{\mathbf{n},m-1}} = -\widehat{\tau}_m, \qquad \text{inside} \qquad \mathbb{C} \setminus \Delta_m,$$

which is equivalent to

$$h - \lim_{n\in\Lambda} \frac{p_{\mathbf{n},m}}{p_{\mathbf{n},m-1}} = -\widehat{\sigma}_m^{-1}, \qquad \text{inside} \qquad \mathbb{C} \setminus \Delta_m,$$

We have proved (3.19) for $j = m - 1$.

For $j = m - 2$, we have shown that there exists a polynomial $w_{\mathbf{n},m-2}, \deg w_{n,m-2} = |\mathbf{n}| - D - \overline{n}_1 - \cdots \overline{n}_{m-3} - 1$, whose zeros lie on Δ_{m-2} such that

$$\frac{\mathcal{A}_{\mathbf{n},m-2}}{w_{\mathbf{n},m-2}} = \frac{p_{\mathbf{n},m-2} + p_{\mathbf{n},m-1}\widehat{\sigma}_{m-1} + p_{\mathbf{n},m}\widehat{\langle\sigma_{m-1},\sigma_m\rangle}}{w_{\mathbf{n},m-2}} \in \mathcal{H}(\mathbb{C} \setminus \Delta_{m-1})$$

and

$$\frac{\mathcal{A}_{\mathbf{n},m-2}}{w_{\mathbf{n},m-2}} = \mathcal{O}\left(\frac{1}{z^{|\mathbf{n}|-D-\overline{n}_1-\cdots-\overline{n}_{m-2}}}\right), \qquad z \to \infty.$$

However, using (4.17) and (4.19), we obtain

$$\frac{p_{\mathbf{n},m-2} + p_{\mathbf{n},m-1}\widehat{\sigma}_{m-1} + p_{\mathbf{n},m}\widehat{\langle\sigma_{m-1},\sigma_m\rangle}}{\widehat{\sigma}_{m-1}} =$$

$$(\ell_{m-1}p_{\mathbf{n},m-2}+p_{\mathbf{n},m-1}+C_1p_{\mathbf{n},m})+p_{\mathbf{n},m-2}\widehat{\tau}_{m-1}-p_{\mathbf{n},m}\langle\tau_{m-1},\widehat{\langle\sigma_m,\sigma_{m-1}\rangle\rangle},$$

where $\deg \ell_{m-1}=1$ and C_1 is a constant. Consequently, $\mathcal{A}_{\mathbf{n},m-2}/(\widehat{\sigma}_{m-1})$ adopts the form of $\mathcal{A}$ in Lemma 3.1.2, $\mathcal{A}_{\mathbf{n},m-2}/(\widehat{\sigma}_{m-1}w_{\mathbf{n},m-2})\in\mathcal{H}(\mathbb{C}\setminus\Delta_{m-1})$, and

$$\frac{\mathcal{A}_{\mathbf{n},m-2}}{\widehat{\sigma}_{m-1}w_{\mathbf{n},m-2}}=\mathcal{O}\left(\frac{1}{z^{|\mathbf{n}|-D-\overline{n}_1-\cdots-\overline{n}_{m-2}-1}}\right),\qquad z\to\infty. \tag{3.23}$$

From Lemma 3.1.2 it follows that for $\nu=0,\ldots,|\mathbf{n}|-D-\overline{n}_1-\cdots-\overline{n}_{m-2}-3$

$$\int_{\Delta_{m-1}}x^\nu\left(p_{\mathbf{n},m-2}(x)-p_{\mathbf{n},m}(x)\widehat{\langle\sigma_m,\sigma_{m-1}\rangle}(x)\right)\frac{d\tau_{m-1}(x)}{w_{\mathbf{n},m-2}(x)}=0.$$

Therefore, $p_{\mathbf{n},m-2}-p_{\mathbf{n},m}\widehat{\langle\sigma_m,\sigma_{m-1}\rangle}\in\mathcal{H}(\mathbb{C}\setminus\Delta_m)$ must have at least $|\mathbf{n}|-D-\overline{n}_1-\cdots-\overline{n}_{m-2}-2$ sign changes on Δ_{m-1}. This means that there exists a polynomial $w^*_{\mathbf{n},m-2}$, $\deg w^*_{\mathbf{n},m-2}=|\mathbf{n}|-D-\overline{n}_1-\cdots-\overline{n}_{m-2}-2$, whose zeros are simple and lie on Δ_{m-1} such that

$$\frac{p_{\mathbf{n},m-2}-p_{\mathbf{n},m}\widehat{\langle\sigma_m,\sigma_{m-1}\rangle}}{w^*_{\mathbf{n},m-2}}\in\mathcal{H}(\mathbb{C}\setminus\Delta_m)$$

and

$$\frac{p_{\mathbf{n},m-2}-p_{\mathbf{n},m}\widehat{\langle\sigma_m,\sigma_{m-1}\rangle}}{w^*_{\mathbf{n},m-2}}=\mathcal{O}\left(\frac{1}{z^{|\mathbf{n}|-D-\overline{n}_1-\cdots-\overline{n}_{m-3}-2\overline{n}_{m-2}-1}}\right).$$

Due to (3.14), this implies that $(p_{\mathbf{n},m-2}/p_{\mathbf{n},m}), n\in\Lambda$, is a sequence of incomplete diagonal Padé approximants of $\widehat{\langle\sigma_m,\sigma_{m-1}\rangle}$ and by Theorem 3.1.5 and Lemma 3.2.2 we obtain its convergence in Hausdorff content to $\widehat{\langle\sigma_m,\sigma_{m-1}\rangle}$. To prove the other part in (3.19), we divide by $\widehat{\langle\sigma_m,\sigma_{m-1}\rangle}(z)$ use (4.17) and proceed as we did in the case $j=m$.

Let us prove (3.19) in general. Fix $j\in\{0,\ldots,m-3\}$ (for $j=m-2,m-1$ it's been proved). Having in mind (3.22) we need to reduce $\mathcal{A}_{\mathbf{n},j}$ so as to eliminate all $p_{\mathbf{n},k}, k=j+1,\ldots,m-1$. We start out eliminating $p_{\mathbf{n},j+1}$. Consider the ratio $\mathcal{A}_{\mathbf{n},j}/\widehat{\sigma}_{j+1}$. Using (4.17) and (4.19) we obtain

$$\frac{\mathcal{A}_{\mathbf{n},j}}{\widehat{\sigma}_{j+1}}=\left(\ell_{j+1}p_{\mathbf{n},j}+\sum_{k=j+1}^m\frac{|s_{j+1,k}|}{|\sigma_{j+1}|}p_{\mathbf{n},j+1}\right)+p_{\mathbf{n},j}\widehat{\tau}_{j+1}-\sum_{k=j+2}^m p_{\mathbf{n},k}\langle\tau_{j+1},\widehat{\langle s_{j+2,k},\sigma_{j+1}\rangle\rangle},$$

and $\mathcal{A}_{\mathbf{n},j}/(\widehat{\sigma}_{j+1})$ has the form of $\mathcal{A}$ in Lemma 3.1.2, where $\mathcal{A}_{\mathbf{n},j}/(\widehat{\sigma}_{j+1}w_{\mathbf{n},j})\in\mathcal{H}(\mathbb{C}\setminus\Delta_{j+1})$, and

$$\frac{\mathcal{A}_{\mathbf{n},j}}{\widehat{\sigma}_{j+1}w_{\mathbf{n},j}}\in\mathcal{O}\left(\frac{1}{z^{|\mathbf{n}|-D-\overline{n}_1-\cdots-\overline{n}_j-1}}\right),\qquad z\to\infty.$$

From Lemma 3.1.2, we obtain that for $\nu = 0, \ldots, |\mathbf{n}| - D - \overline{n}_1 - \cdots - \overline{n}_j - 3$

$$0 = \int_{\Delta_{j+1}} x^\nu \left(p_{\mathbf{n},j}(x) - \sum_{k=j+2}^{m} p_{\mathbf{n},k} \widehat{\langle s_{j+2,k}, \sigma_{j+1} \rangle}(x) \right) \frac{d\tau_{j+1}(x)}{w_{\mathbf{n},j}(x)}$$

which implies that the function in parenthesis under the integral sign has at least $|\mathbf{n}| - D - \overline{n}_1 - \cdots - \overline{n}_j - 2$ sign changes on Δ_{j+1}. In turn, it follows that there exists a polynomial $\widetilde{w}_{\mathbf{n},j+1}$, $\deg \widetilde{w}_{\mathbf{n},j+1} = |\mathbf{n}| - D - \overline{n}_1 - \cdots - \overline{n}_j - 2$, whose zeros are simple and lie on Δ_{j+1} such that

$$\frac{p_{\mathbf{n},j} - \sum_{k=j+2}^{m} p_{\mathbf{n},k} \widehat{\langle s_{j+2,k}, \sigma_{j+1} \rangle}}{\widetilde{w}_{\mathbf{n},j+1}} \in \mathcal{H}(\mathbb{C} \setminus \Delta_{j+2})$$

and

$$\frac{p_{\mathbf{n},j} - \sum_{k=j+2}^{m} p_{\mathbf{n},k} \widehat{\langle s_{j+2,k}, \sigma_{j+1} \rangle}}{\widetilde{w}_{\mathbf{n},j+1}} = \mathcal{O}\left(\frac{1}{z^{|\mathbf{n}| - D - \overline{n}_1 - \cdots - \overline{n}_{j-1} - 2\overline{n}_j - 1}} \right), \qquad z \to \infty.$$

Notice that $p_{\mathbf{n},j+1}$ has been eliminated and that

$$\langle s_{j+2,k}, \sigma_{j+1} \rangle = \langle \langle \sigma_{j+2}, \sigma_{j+1} \rangle, \sigma_{j+3}, \ldots, \sigma_k \rangle, \qquad k = j+3, \ldots, m.$$

Now we must do away with $p_{\mathbf{n},j+2}$ in $p_{\mathbf{n},j} - \sum_{k=j+2}^{m} p_{\mathbf{n},k} \widehat{\langle s_{j+2,k}, \sigma_{j+1} \rangle}$ (in case that $j + 2 < m$). To this end, we consider the ratio

$$\frac{p_{\mathbf{n},j} - \sum_{k=j+2}^{m} p_{\mathbf{n},k} \widehat{\langle s_{j+2,k}, \sigma_{j+1} \rangle}}{\widehat{\langle \sigma_{j+2}, \sigma_{j+1} \rangle}}$$

and repeat the arguments employed above with $\mathcal{A}_{\mathbf{n},j}$. After $m - j - 2$ reductions obtained applying consecutively Lemma 3.1.2, we find that there exists a polynomial which we denote $w^*_{\mathbf{n},j}$, $\deg w^*_{\mathbf{n},j} = |\mathbf{n}| - D - \overline{n}_1 - \cdots - \overline{n}_{j-1} - (m - j - 1)\overline{n}_j - 2$ whose zeros are simple and lie on Δ_{m-1} such that

$$\frac{p_{\mathbf{n},j} - (-1)^{m-j} p_{\mathbf{n},m} \widehat{\langle \sigma_m, \ldots, \sigma_{j+1} \rangle}}{w^*_{\mathbf{n},j}} \in \mathcal{H}(\mathbb{C} \setminus \Delta_m)$$

and

$$\frac{p_{\mathbf{n},j} - (-1)^{m-j} p_{\mathbf{n},m} \widehat{\langle \sigma_m, \ldots, \sigma_{j+1} \rangle}}{w^*_{\mathbf{n},j}} = \mathcal{O}\left(\frac{1}{z^{|\mathbf{n}| - \overline{n}_1 - \cdots - \overline{n}_{j-1} - (m-j)\overline{n}_j - 1}} \right), \qquad z \to \infty.$$

Dividing by $(-1)^{m-j}\langle \sigma_m,\ldots,\sigma_{j+1}\widehat{\rangle}$, from here it also follows that

$$\frac{p_{\mathbf{n},j}(-1)^{m-j}\langle \sigma_m,\ldots,\sigma_{j+1}\widehat{\rangle}^{-1} - p_{\mathbf{n},m}}{w^*_{\mathbf{n},j}} \in \mathcal{H}(\mathbb{C}\setminus\Delta_m)$$

and

$$\frac{p_{\mathbf{n},j}(-1)^{m-j}\langle \sigma_m,\ldots,\sigma_{j+1}\widehat{\rangle}^{-1} - p_{\mathbf{n},m}}{w^*_{\mathbf{n},j}} = \mathcal{O}\left(\frac{1}{z^{|\mathbf{n}|-D-\overline{n}_1-\cdots-\overline{n}_{j-1}-(m-j)\overline{n}_j-2}}\right), \quad z\to\infty.$$

On account of (3.14), these relations imply that $(p_{\mathbf{n},j}/p_{\mathbf{n},m}), \mathbf{n}\in\Lambda$, is a sequence of incomplete diagonal multi-point Padé approximants of $(-1)^{m-j}\langle \sigma_m,\ldots,\sigma_{j+1}\widehat{\rangle}$ and $(p_{\mathbf{n},m}/p_{\mathbf{n},j}), \mathbf{n}\in\Lambda$, is a sequence of incomplete diagonal multi-point Padé approximants of $(-1)^{m-j}\langle \sigma_m,\ldots,\sigma_{j+1}\widehat{\rangle}^{-1}$. Since $\langle \sigma_m,\ldots,\sigma_{j+1}\widehat{\rangle}^{-1} = \widehat{\tau}_{m,j+1}+\ell_{m,j+1}, \deg \ell_{m,j+1}=1$, from Theorem 3.1.5 and Lemma 3.2.2 we obtain (3.19). □

Now, from convergence in 1-Hausdorff content we wish to deduce uniform convergence on compact subsets. In this connection, [37, Lemma 1] of A.A. Gonchar is the key. We state it for convenience of the reader.

Lemma 3.2.4. *Suppose that h-$\lim_{n\to\infty}\varphi_n=\varphi$ inside D. Then the following assertions hold true:*

i) If the functions φ_n, $n\in\mathbb{N}$, are holomorphic in D, then the sequence $\{\varphi_n\}$ converges uniformly on compact subsets of D and φ is holomorphic in D (more precisely, it is equal to a holomorphic function in D except on a set of h-content zero).

ii) If each of the functions φ_n is meromorphic in D and has no more than $k<+\infty$ poles in this domain, then the limit function φ is (again except on a set of h-content zero) also meromorphic and has no more than k poles in D.

iii) If each function φ_n is meromorphic and has no more than $k<+\infty$ poles in D and the function φ is meromorphic and has exactly k poles in D, then all φ_n, $n\geq N$, also have k poles in D; the poles of φ_n tend to the poles $z_1,,\ldots,z_k$ of φ (taking account of their orders) and the sequence $\{\varphi_n\}$ tends to φ uniformly on compact subsets of the domain $D'=D\setminus\{z_1,,\ldots,z_k\}$.

Remark 3.2.5. *The statement of Gonchar's lemma remains valid if we change in the hypothesis the convergence in 1-dimensional Hausdorff content by the convergence in logaritmic capacity.*

In the case of decreasing components in $\mathbf{n}$, we saw that all the zeros of $a_{\mathbf{n},m}$ lie in Δ_m and [37, Lemma 1] would allow us to derive immediately uniform convergence on each compact subset of $\mathbb{C} \setminus \Delta_m$ from the convergence in Hausdorff content. For other configurations of the components of **n** we have to work a little harder.

Proof of Theorem 3.2.1. Let $\overline{\jmath}$ be the last component of $(n_0, \ldots, n_m)$ such that $n_{\overline{\jmath}} = \min_{j=0,\ldots,m}(n_j)$. Let us prove that $\deg a_{\mathbf{n},\overline{\jmath}} = n_{\overline{\jmath}} - 1$, that all its zeros are simple and lie in $\overset{\circ}{\Delta}_m$.

From [33, Theorem 3.2] (see also [32, Theorem 1.3]) we know that there exists a permutation λ of $(0, \ldots, m)$ which reorders the components of $(n_0, n_1, \ldots, n_m)$ decreasingly, $n_{\lambda(0)} \geq \cdots \geq n_{\lambda(m)}$, and an associated Nikishin system $(r_{1,1}, \ldots, r_{1,m}) = \mathcal{N}(\rho_1, \ldots, \rho_m)$ such that

$$\mathcal{A}_{\mathbf{n},0} = (q_{\mathbf{n},0} + \sum_{k=1}^{m} q_{\mathbf{n},k}\widehat{r}_{1,k})\widehat{s}_{1,\lambda(0)}, \qquad \deg q_{\mathbf{n},k} \leq n_{\lambda(k)} - 1, \qquad k = 0, \ldots, m.$$

The permutation may be taken so that for all $0 \leq j < k \leq n$ with $n_j = n_k$ then also $\lambda(j) < \lambda(k)$. In this case, see formulas (31) in the proof of [32, Lemma 2.3], it follows that $q_{\mathbf{n},m} = \pm a_{\mathbf{n},\overline{\jmath}}$. Reasoning with $q_{\mathbf{n},0} + \sum_{k=1}^{m} q_{\mathbf{n},k}\widehat{r}_{1,k}$ as we did with $\mathcal{A}_{\mathbf{n},0}$ we obtain that $\deg q_{\mathbf{n},m} = n_{\lambda(m)} - 1$ and that its zeros are all simple and lie in $\overset{\circ}{\Delta}_m$. However, $n_{\lambda(m)} = n_{\overline{\jmath}}$ and $q_{\mathbf{n},m} = \pm a_{\mathbf{n},\overline{\jmath}}$ so the statement holds.

The index $\overline{\jmath}$ as defined above may depend on the multi-index $\mathbf{n} \in \Lambda$. Given $\overline{\jmath} \in \{0, \ldots, m\}$, let us denote by $\Lambda(\overline{\jmath})$ the set of all $\mathbf{n} \in \Lambda$ such that $\overline{\jmath}$ is the last component of $(n_0, \ldots, n_m)$ such that $n_{\overline{\jmath}} = \min_{j=0,\ldots,m}(n_j)$. Fix $\overline{\jmath}$ and suppose that $\Lambda(\overline{\jmath})$ contains infinitely many multi-indices. If $\overline{\jmath} = m$, then [37, Lemma 1] and the first limit in (3.19) imply that

$$\lim_{n \in \Lambda(m)} \frac{a_{\mathbf{n},j}}{a_{\mathbf{n},m}} = (-1)^{m-j}\widehat{s}_{m,j+1}, \qquad j = 0, \ldots, m-1,$$

uniformly on each compact subset of $\mathbb{C} \setminus \Delta_m$, as needed.

Assume that $\overline{\jmath} \in \{0, \ldots, m-1\}$. Since all the zeros of $a_{\mathbf{n},\overline{\jmath}}$ lie in $\overset{\circ}{\Delta}_m$, using [37, Lemma 1] and the second limit in (3.19) for $j = \overline{\jmath}$, we obtain that

$$\lim_{\mathbf{n} \in \Lambda(\overline{\jmath})} \frac{a_{\mathbf{n},m}}{a_{\mathbf{n},\overline{\jmath}}} = \frac{1}{(-1)^{m-\overline{\jmath}}\widehat{s}_{m,\overline{\jmath}+1}}, \tag{3.24}$$

uniformly on each compact subset of $\mathbb{C} \setminus \Delta_m$. The function on the right hand side of (3.24) is holomorphic and never zero on $\mathbb{C} \setminus \Delta_m$ and the approximating functions are holomorphic on $\mathbb{C} \setminus \Delta_m$. Using Rouche's theorem it readily follows that on any compact subset $K \subset \mathbb{C} \setminus \Delta_m$ for all sufficiently large $|\mathbf{n}|, n \in \Lambda(\overline{\jmath})$, the polynomials $a_{\mathbf{n},m}$ have no zero on K. This is true for any $\overline{\jmath} \in \{0, \ldots, m\}$ such that $\Lambda(\overline{\jmath})$ contains infinitely many multi-indices. Therefore, the only accumulation points of the zeros of the polynomials $a_{\mathbf{n},m}$ are in $\Delta_m \cup \{\infty\}$.

Hence, on any bounded region D such that $\overline{D} \subset \mathbb{C} \setminus \Delta_m$ for each fixed $j = 0, \ldots, m-1$, and all sufficiently large $|\mathbf{n}|, \mathbf{n} \in \Lambda$, we have that $a_{\mathbf{n},j}/a_{\mathbf{n},m} \in \mathcal{H}(D)$. From [37, Lemma 1] and the first part of (3.19) it follows that

$$\lim_{\mathbf{n} \in \Lambda} \frac{a_{\mathbf{n},j}}{a_{\mathbf{n},m}} = (-1)^{m-j} \widehat{s}_{m,j+1}, \qquad j = 0, \ldots, m-1, \tag{3.25}$$

uniformly on each compact subset of D. Since D was chosen arbitrarily, as long as $\overline{D} \subset \mathbb{C} \setminus \Delta_m$, it follows that the convergence is uniform on each compact subset of $\mathbb{C} \setminus \Delta_m$ and we have (3.15).

Now,

$$\frac{\mathcal{A}_{\mathbf{n},j}}{a_{\mathbf{n},m}} = \frac{a_{\mathbf{n},j}}{a_{\mathbf{n},m}} + \sum_{k=j+1}^{m-1} \frac{a_{\mathbf{n},k}}{a_{\mathbf{n},m}} \widehat{s}_{j+1,k} + \widehat{s}_{j+1,m}.$$

According to formula (17) in [32, Lemma 2.9]

$$0 \equiv (-1)^{m-j} \widehat{s}_{m,j+1} + \sum_{k=j+1}^{m-1} (-1)^{m-k} \widehat{s}_{m,k+1} \widehat{s}_{j+1,k} + \widehat{s}_{j+1,m}, \quad z \in \mathbb{C} \setminus (\Delta_{j+1} \cup \Delta_m).$$

Deleting one expression from the other we have that

$$\frac{\mathcal{A}_{\mathbf{n},j}}{a_{\mathbf{n},m}} = \left(\frac{a_{\mathbf{n},j}}{a_{\mathbf{n},m}} - (-1)^{m-j} \widehat{s}_{m,j+1} \right) + \sum_{k=j+1}^{m-1} \left(\frac{a_{\mathbf{n},k}}{a_{\mathbf{n},m}} - (-1)^{m-k} \widehat{s}_{m,k+1} \right) \widehat{s}_{j+1,k} \tag{3.26}$$

Consequently, for each $j = 0, \ldots, m-1$, from (3.15) we obtain

$$\lim_{\mathbf{n} \in \Lambda} \frac{\mathcal{A}_{\mathbf{n},j}}{a_{\mathbf{n},m}} = 0$$

uniformly on each compact subset of $\mathbb{C} \setminus (\Delta_{j+1} \cup \Delta_m)$ which is (3.16). $\square$

Suppose that Δ_m is bounded. Let Γ be a positively oriented closed simple Jordan curve that surrounds Δ_m. Define $\kappa_{\mathbf{n},j}(\Gamma), j = 0, \ldots, m$ to be the number of zeros of $a_{\mathbf{n},j}$ outside Γ. As above, given $\overline{\jmath} \in \{0, \ldots, m\}$, let us denote by $\Lambda(\overline{\jmath})$ the set of all $\mathbf{n} \in \Lambda$ such that $\overline{\jmath}$ is the last component of $(n_0, \ldots, n_m)$ such that $n_{\overline{\jmath}} = \min_{j=0,\ldots,m}(n_j)$.

Corollary 3.2.6. *Suppose that the assumptions of Theorem 3.2.1 hold and Δ_m is bounded. Then for all sufficiently large $|\mathbf{n}|, \mathbf{n} \in \Lambda(\overline{\jmath})$,*

$$\kappa_{\mathbf{n},j}(\Gamma) = \begin{cases} n_j - n_{\overline{\jmath}}\,, & j = 0, \ldots, m-1, \\ n_m - n_{\overline{\jmath}} - 1, & j = m. \end{cases} \tag{3.27}$$

The rest of the zeros of the polynomials $a_{\mathbf{n},j}$ accumulate (or lie) on Δ_m.

Note that by Theorem 2.1.12 we know that the zeros of the linear form $\mathcal{A}_{\mathbf{n},0}$ are simple and lie on Δ_1, this corollary completes the information about the behaviour of the zeros of $\mathcal{A}_{\mathbf{n},0}$.

Proof. Fix $\overline{\jmath} \in \{0, \ldots, m-1\}$. Assume that $\Lambda(\overline{\jmath})$ contains infinitely many multi-indices. Using the argument principle and (3.24) it follows that

$$\lim_{\mathbf{n}\in\Lambda(\overline{\jmath})} \frac{1}{2\pi i}\int_\Gamma \frac{(a_{\mathbf{n},m}/a_{\mathbf{n},\overline{\jmath}})'(z)}{(a_{\mathbf{n},m}/a_{\mathbf{n},\overline{\jmath}})(z)}dz = \frac{1}{2\pi i}\int_\Gamma \frac{(1/\widehat{s}_{m,\overline{\jmath}+1})'(z)}{(1/\widehat{s}_{m,\overline{\jmath}+1})(z)}dz = 1,$$

because $1/\widehat{s}_{m,\overline{\jmath}+1}$ has one pole and no zeros outside Γ (counting the point ∞). Let $\kappa_{\mathbf{n},j}(\Gamma)$ denote the number of zeros of $a_{\mathbf{n},j}$ outside Γ. Recall that $\deg a_{\mathbf{n},j} = n_j - 1, j = 0, \ldots, m$ and that all the zeros of $a_{\mathbf{n},\overline{\jmath}}$ lie on Δ_m. Then, for all sufficiently large $|\mathbf{n}|, \mathbf{n} \in \Lambda(\overline{\jmath})$,

$$(n_m - 1) - (n_{\overline{\jmath}} - 1) - \kappa_{\mathbf{n},m}(\Gamma) = 1.$$

Consequently,

$$\kappa_{\mathbf{n},m}(\Gamma) = n_m - n_{\overline{\jmath}} - 1, \qquad \mathbf{n} \in \Lambda(\overline{\jmath}). \tag{3.28}$$

Analogously, from (3.25), for $j = 0, \ldots, m-1$, we obtain

$$\lim_{\mathbf{n}\in\Lambda} \frac{1}{2\pi i}\int_\Gamma \frac{(a_{\mathbf{n},j}/a_{\mathbf{n},m})'(z)}{(a_{\mathbf{n},j}/a_{\mathbf{n},m})(z)}dz = \frac{1}{2\pi i}\int_\Gamma \frac{\widehat{s}'_{m,j+1}(z)}{\widehat{s}_{m,j+1}(z)}dz = -1.$$

Therefore, for all sufficiently large $|\mathbf{n}|, \mathbf{n} \in \Lambda$,

$$n_j - n_m + \kappa_{\mathbf{n},m}(\Gamma) - \kappa_{\mathbf{n},j}(\Gamma) = -1, \qquad j = 0, \ldots, m-1,$$

which together with (3.28) gives (3.27). The last statement follows from the fact that the only accumulation points of the zeros of the $a_{\mathbf{n},j}$ are in $\Delta_m \cup \{\infty\}$. $\square$

Remark 3.2.7. *The thesis of Theorem 3.2.1 remains valid if in place of* (3.14) *we require that*

$$n_j = \frac{|\mathbf{n}|}{m} + o(|\mathbf{n}|), \qquad |\mathbf{n}| \to \infty, \qquad j = 1, \ldots, m. \tag{3.29}$$

To prove this we need an improved version of Lemma 3.2.2 in which the parameter ℓ in b) depends on n but $\ell(n) = o(n), n \to \infty$. The proof of Lemma 2 in [15] admits this variation with some additional technical difficulties which were in part resolved in the proof of [31, Corollary 1].

Remark 3.2.8. *If either Δ_m or Δ_{m-1} is a compact set and $\Delta_{m-1} \cap \Delta_m = \emptyset$, it is not difficult to show that convergence takes place in* (3.15) *and* (3.16) *with geometric rate. More precisely, for $j = 0, \ldots, m-1$, and $K \subset \mathbb{C} \setminus \Delta_m$, we have*

$$\limsup_{\mathbf{n} \in \Lambda} \left\| \frac{a_{\mathbf{n},j}}{a_{\mathbf{n},m}} - (-1)^{m-j} \widehat{s}_{m,j+1} \right\|_K^{1/|\mathbf{n}|} = \delta_j < 1. \tag{3.30}$$

For $j = 0, \ldots, m-1$, and $K \subset \mathbb{C} \setminus (\Delta_{j+1} \cup \Delta_m)$

$$\limsup_{\mathbf{n} \in \Lambda} \left\| \frac{\mathcal{A}_{\mathbf{n},j}}{a_{\mathbf{n},m}} \right\|_K^{1/|\mathbf{n}|} \le \max\{\delta_k : j \le k \le m-1\} < 1. \tag{3.31}$$

The second relation trivially follows from the first and (4.11). *The proof of the first is similar to that of [31, Corollary 1]. It is based on the fact that the number of interpolation point on Δ_{m-1} is of the same order as $|\mathbf{n}| \to \infty$, and that the distance from Δ_m to Δ_{m-1} is positive. Relations* (3.30) *and* (3.31) *are also valid if* (3.14) *is replaced with* (3.29).

Asymptotically, (3.29) *still means that the components of* $\mathbf{n}$ *are equally valued. One can relax* (3.29) *when, for example, the generating measures are regular in the sense of [74, Chapter 3]. Then the exact asymptotics of* (3.30) *and* (3.31) *can be given (see [29, Theorem 5.1, Corollary 5.3], [60], [61, Chapter 5, Section 7] and [64, Theorem 1]) requiring basically, that*

$$\frac{n_j}{|\mathbf{n}|} = \theta_j + \mathcal{O}(1), \quad 0 < \theta_j < 1, \quad j = 1, \ldots, m$$

Remark 3.2.9. *The previous results can be applied to mixed type Hermite-Padé approximation. Let $S^1 = \mathcal{N}(\sigma_0^1, \ldots, \sigma_{m_1}^1), S^2 = \mathcal{N}(\sigma_0^2, \ldots, \sigma_{m_2}^2), \sigma_0^1 = \sigma_0^2$ be given. Fix $\mathbf{n}_1 = (n_{1,0}, n_{1,1}, \ldots, n_{1,m_1}) \in \mathbb{Z}_+^{m_1+1}$ and $\mathbf{n}_2 = (n_{2,0}, n_{2,1}, \ldots, n_{2,m_2}) \in \mathbb{Z}_+^{m_2+1}, |\mathbf{n}_2| = |\mathbf{n}_1| - 1$.*

Let us introduce the row vectors

$$\mathbf{v} = (1, \widehat{s}^2_{1,1}, \ldots, \widehat{s}^2_{1,m_2}), \qquad \mathbf{u} = (1, \widehat{s}^1_{1,1}, \ldots, \widehat{s}^1_{1,m_1})$$

and the $(m_2+1) \times (m_1+1)$ *dimensional measure matrix*

$$d\mathbf{S} = \mathbf{v}^T \mathbf{u} d\sigma^2_0.$$

Denote by $(a_{\mathbf{n},0}, \ldots, a_{\mathbf{n},m_1})$, *the mixed type Hermite-Padé approximants relative to* $\widehat{\mathbf{S}}$ *where*

$$\widehat{\mathbf{S}}(z) = \int \frac{d\mathbf{S}(\mathbf{x})}{z-x}$$

understanding that integration is carried out entry by entry on the matrix $\mathbf{S}$. *This implies*

$$\int \left(b_{\mathbf{n},0}(x) + \sum_{j=1}^{m_2} b_{\mathbf{n},j}(x) \widehat{s}^2_{1,j}(x) \right) \mathcal{A}_{\mathbf{n},0}(x) d\sigma^2_0(x) = 0, \tag{3.32}$$

where $\mathcal{A}_{\mathbf{n},0} = a_{\mathbf{n},0} + \sum_{k=1}^{m_1} a_{\mathbf{n},k} \hat{s}_{1,k}$ *and* $b_{\mathbf{n},j}$ *are arbitrary polynomials with* $\deg b_{\mathbf{n},j} \leq n_{2,j} - 1$, $j = 0, \ldots, m_2$.

Then $\mathcal{A}_{\mathbf{n},0}$ *has exactly* $|\mathbf{n_2}|$ *zeros in* $\mathbb{C} \setminus \Delta^1_1$ *they are all simple and lie in* $\overset{\circ}{\Delta}{}^1_0$ *(see Theorem 2.1.9). Here* $\Delta^1_0 = Co(\mathrm{supp}(\sigma^1_0))$ *and* $\Delta^1_1 = Co(\mathrm{supp}(\sigma^1_1))$. *Denote by* $w_{\mathbf{n}}$ *the monic polynomial whose zeros are the zeros of* $\mathcal{A}_{\mathbf{n},0}$ *on* $\mathbb{C} \setminus \Delta^1_1$. *Therefore,* $(a_{\mathbf{n},0}, \ldots, a_{\mathbf{n},m})$ *is a type I multi-point Hermite-Padé approximation of* $(\widehat{s}_{1,1}, \ldots, \widehat{s}_{1,m_1})$ *with respect to* $w_{\mathbf{n}}$ *and the Theorem 3.2.1 may be applied.*

In the process of writing this thesis we realized that the following result which extends Theorem 3.2.1 is valid. The proof is basically the same so we omit it.

Consider the system of meromorphic functions of the form $\mathbf{f} = (f_1, \ldots, f_m) = \widehat{\mathbf{s}} + \mathbf{r}$, where

$$f_j(z) = \widehat{s}_{1,j}(z) + r_j(z), \qquad j = 1, \ldots, m, \tag{3.33}$$

here, $\mathbf{r} = (r_1, \ldots, r_m) = \left(\frac{v_1}{t_1}, \ldots, \frac{v_m}{t_m} \right)$, is a vector rational fractions with real coefficients such that $\deg t_j = d_j$ and $\deg v_j < d_j$, for every $j = 1, \ldots, m$. We assume that $v_j/t_j, j = 1, \ldots, m$ is irreducible and $\mathbf{s} = (s_{1,1} \ldots, s_{1,m})$ is a Nikshin system.

Theorem 3.2.10. *Let $\Lambda \subset \mathbb{Z}_+^m$ be an infinite sequence of distinct muti-indices. Consider the corresponding sequence $(a_{\mathbf{n},0}, \ldots, a_{\mathbf{n},m})$, $\mathbf{n} \in \Lambda$, of type I Hermite-Padé approximants of* $\mathbf{f}$ *. Assume that the rational functions $r_1, \ldots r_m$ have real coefficients and their poles lie in $\mathbb{C} \setminus (\Delta_1 \cup \Delta_m)$, for $j \neq k$ the poles of r_j and r_k are distinct. Assume that* (3.14) *takes place and that either Δ_{m-1} is bounded away from Δ_m or σ_m satisfies Carleman's condition. Then, for $j = 1, \ldots, m-1$*

$$\lim_{\mathbf{n}\in\Lambda} \frac{a_{\mathbf{n},j}}{a_{\mathbf{n},m}} = (-1)^{m-j} \widehat{s}_{m,j+1}, \tag{3.34}$$

and

$$\lim_{\mathbf{n}\in\Lambda} \frac{a_{\mathbf{n},0}}{a_{\mathbf{n},m}} = (-1)^m \widehat{s}_{m,1} - \sum_{j=1}^{m-1} (-1)^{m-j} r_j \widehat{s}_{m,j+1} + r_m \tag{3.35}$$

uniformly on each compact subset K contained in $(\mathbb{C} \setminus \Delta_m)'$ the set obtained deleting from $\mathbb{C} \setminus \Delta_m$ the poles of all the r_j.

Notice that the rational fractions $(r_1, \ldots, r_m)$ do not play any role in the expression of the limit of $(\frac{a_{\mathbf{n},1}}{a_{\mathbf{n},m}}, \ldots, \frac{a_{\mathbf{n},m-1}}{a_{\mathbf{n},m}})$. On the other hand, all the information of $(r_1, \ldots, r_m)$ is contained in the expression of the limit of $\frac{a_{\mathbf{n},0}}{a_{\mathbf{n},m}}$.

Chapter 4

Hermite-Padé approximants for perturbed Nikishin systems

«CONTENTS»

- Convergence of type I Hermite-Padé approximants for certain systems of meromorphic functions.
- AT property for polynomial modification of Nikishin systems.
- Convergence of type II Hermite-Padé approximants for certain systems of meromorphic functions.

SEQUENCES of type II Hermite-Padé approximants of systems of meromorphic functions are considered. These systems are constructed adding vectors of rational fractions to Nikishin systems of functions. In this chapter, we give general sufficient conditions for their convergence. The convergence of type I Hermite-Padé approximation for perturbed Nikishin systems is also analyzed, here these system are constructed multiplying the functions of a Nikishin system by polynomials. We obtain extensions of Markov's and Stieltjes' theorem.

Recall that combinig Stieltjes' theorem and Carleman's condition (see [17]) we can state that if $s \in \mathcal{M}(\mathbb{R})$ satisfies

$$\sum_{n\geq 0} |c_n|^{-1/2n} = \infty, \tag{4.1}$$

then

$$\lim_{n\to\infty} \frac{P_n}{Q_n}(z) = \widehat{s}(z), \tag{4.2}$$

uniformly on each compact subset of (inside) $\mathbb{C} \setminus \Delta$.

In an attempt to extend Markov's theorem to a general class of meromorphic functions, A.A. Gonchar considered functions of the form $\widehat{s}+r$ where r is a rational function whose poles lie in $\mathbb{C} \setminus \Delta$. In [36], he proved that if Δ is a bounded interval and $s' > 0$ a.e. on Δ, then (4.2) takes place showing, additionally, that each pole of r in $\mathbb{C} \setminus \Delta$ "attracts" as many zeros of Q_n as its order and the remaining zeros of Q_n accumulate on Δ as $n \to \infty$. Later, in [65] E.A. Rakhmanov obtained a full extension of Markov's theorem when r has real coefficients and proved that if r has complex coefficients then such a result is not possible without extra assumptions on the measure s. The case of unbounded Δ was solved in [51], when r has real coefficients, and [53], when r has complex coefficients.

The corresponding problem in the vector case practically has not been considered. To our knowledge the only results in this direction are contained in [16] for systems of 2 functions. We complement and extend this study for arbitrary Nikishin systems.

Consider a vector of rational functions $\mathbf{r} = (r_1, \ldots, r_m) = \left(\frac{v_1}{t_1}, \ldots, \frac{v_m}{t_m}\right)$, such that $\deg t_j = d_j$ and $\deg v_j < d_j$, for every $j = 1, \ldots, m$. We assume that v_j/t_j, $j = 1, \ldots, m$ is irreducible. We consider systems of meromorphic functions of the form $\mathbf{f} = (f_1, \ldots, f_m) = \widehat{\mathbf{s}} + \mathbf{r}$, where

$$f_j(z) = \widehat{s}_{1,j}(z) + r_j(z), \qquad j = 1, \ldots, m. \tag{4.3}$$

where $\mathbf{s} = (s_{1,1}, \ldots, s_{1,m}) = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$.

Our aim is to study the convergence of diagonal type II Hermite-Padé approximants to meromorphic functions of the form $\mathbf{f} = \widehat{\mathbf{s}} + \mathbf{r}$. In this study we have to deal with a special type of incomplete type I multi-point Hermite Padé approximant for which we extend the results of Chapter 3.

This chapter is organized as follows. Section 4.1 contains the results related to type I multi-point Hermite Padé approximants, here we show that certain polynomial modification of Nikishin systems form AT systems and prove a Markov theorem for type I Hermite-Padé approximants with respect to such perturbed Nikishin systems. In Section 4.2 we prove the convergence of multi-point type II Hermite Padé approximants with respect to systems of the form (4.3) and obtain an analogue of Markov's theorem for an arbitray m assuming that $\mathbf{r}$ has real coefficients and some natural restriction on the location and distribution of its poles. Finally the case when the vector of rational fraction $\mathbf{r}$ has complex coefficients is studied in Section 4.3 and convergence in Hausdorff content is obtained.

4.1 Type I Hermite-Padé approximants

In this section $\mathbf{n} := (n_0, \ldots, n_m) \in \mathbb{Z}_+^{m+1} \setminus \{\mathbf{0}\}$, $|\mathbf{n}| = n_0 + \cdots + n_m$, and $N_{\mathbf{n}} = \max\{n_0, n_1 - 1, \ldots, n_m - 1\}$.

We are interested in a special type of incomplete type I multi-point Hermite Padé approximant.

Let $(t_0, t_1, \ldots, t_m), \deg t_j = d_j$, be a vector polynomial with real coefficients and let $D := \sum_{j=0}^m d_j$. Fix $\mathbf{n} = (n_0, \ldots, n_m) \in \mathbb{Z}_+^{m+1}, n_j > D, j = 0, \ldots, m$. Given $(s_{1,1}, \ldots, s_{1,m}) = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$ and a polynomial with real coefficients $w_{\mathbf{n}}, \deg w_{\mathbf{n}} \leq |\mathbf{n}| - D - 1$, whose zeros lie in $\mathbb{R} \setminus \Delta_1$, there exist polynomials $a_{\mathbf{n},0}, a_{\mathbf{n},1}, \ldots, a_{\mathbf{n},m}$, not all identically equal to zero, such that:

i') $\deg a_{\mathbf{n},j} t_j \leq n_j - 1, j = 0, \ldots, m,$
ii') $\frac{a_{\mathbf{n},0}(z)t_0(z) + \sum_{j=1}^m a_{\mathbf{n},j}(z)t_j(z)\hat{s}_{1,j}(z)}{w_{\mathbf{n}}(z)} = \mathcal{O}(1/z^{|\mathbf{n}|-N_{\mathbf{n}}-D}) \in \mathcal{H}(\mathbb{C} \setminus \Delta_1).$

In other words, we use the freedom in the construction of the incomplete type I Hermite-Padé approximants to force the polynomials $a_{\mathbf{n},j} t_j, j = 0, \ldots, m$ to have some predetermined zeros. $(a_{\mathbf{n},0}, \ldots, a_{\mathbf{n},m})$ could be regarded as a type I multi-point Hermite Padé approximant of $(t_1 \widehat{s}_{1,1}/t_0, \ldots, t_m \widehat{s}_{1,m}/t_0)$ with respect to $(n_0 - d_0, \ldots, n_m - d_m)$.

Note that type I Hermite-Padé approximation is contained in this definition. As we did with the type I Hermite-Padé approximants, for these incomplete type I multi-point Hermite Padé approximants we can obtain an analogue of Markov's theorem.

Theorem 4.1.1. *Let $(s_{1,1}, \ldots, s_{1,m}) = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$ and $\Lambda \subset \mathbb{Z}_+^m$ be an infinite sequence of distinct multi-indices. Suppose that the polynomials $t_0, t_1, \ldots t_m$ have no common zeros, and they all lie in $\mathbb{C} \setminus \Delta_m$. Assume that* (3.14) *takes place and either Δ_{m-1} is bounded away from Δ_m or σ_m satisfies* (4.1). *Suppose that for each $\mathbf{n} \in \Lambda$ the polynomials $a_{\mathbf{n},0}, a_{\mathbf{n},1}, \ldots, a_{\mathbf{n},m}$ satisfy i')-ii'). Then, for $j = 0, 1, \ldots m - 1$*

$$\lim_{\mathbf{n} \in \Lambda} \frac{a_{\mathbf{n},j}}{a_{\mathbf{n},m}} = (-1)^{m-j} \frac{t_m}{t_j} \hat{s}_{m,j+1}, \qquad \textit{inside} \qquad (\mathbb{C} \setminus \Delta_m)', \tag{4.4}$$

the set obtained deleting from $\mathbb{C} \setminus \Delta_m$ the zeros of all the polynomials t_j. Additionally, for $j = 0, \ldots, m - 1$, if we denote

$$\mathcal{A}_{\mathbf{n},j}(z) := a_{\mathbf{n},j}(z)t_j(z) + \sum_{k=j+1}^m a_{\mathbf{n},k}(z)t_k(z)\widehat{s}_{j+1,k}(z),$$

then,

$$\lim_{\mathbf{n}\in\Lambda}\frac{\mathcal{A}_{\mathbf{n},j}}{a_{\mathbf{n},m}}=0,\qquad \textit{inside}\qquad (\mathbb{C}\setminus(\Delta_{j+1}\cup\Delta_m))', \tag{4.5}$$

the set obtained deleting from $\mathbb{C}\setminus(\Delta_{j+1}\cup\Delta_m)$ the zeros of all the polynomials t_j.

Moreover there exists a constant C_1, independent of Λ, such that for each $j=0,\ldots,m$ and $\mathbf{n}\in\Lambda$, the polynomials $a_{\mathbf{n},j}$ have at least $(|\mathbf{n}|/(m+1))-C_1$ zeros in $\overset{\circ}{\Delta}_m$. Fix $j,k=0,\ldots,m$. Let ζ be a zero of $t_k, k\neq j$, of multiplicity κ. Then, for each $\varepsilon>0$ sufficiently small there exists an N such that for all $\mathbf{n}\in\Lambda, |\mathbf{n}|>N$, $a_{\mathbf{n},j}$ has exactly κ zeros in $\{z:|z-\zeta|<\varepsilon\}$. The remaining zeros of $a_{\mathbf{n},j}$ either lie on Δ_m or accumulate on $\Delta_m\cup\{\infty\}$ as $|\mathbf{n}|\to\infty$.

Proof. The lower bound on the number of zeros in $\overset{\circ}{\Delta}_m$ is taken directly from Lemma 3.2.3. Let $\bar{j}$ be the last component of $(n_0,\ldots,n_m)$ such that $n_{\bar{j}}=\min_{j=0,\ldots,m}(n_j)$. Let us prove that $a_{\mathbf{n},\bar{j}}$ has at least $n_{\bar{j}}-D-1$ zeros in $\overset{\circ}{\Delta}_m$. That is, for this component we want a more precise lower bound than the one given in the statement of the theorem.

From [32, Theorem 1.3] (see also [33, Theorem 3.2]), we know that there exists a permutation λ of $(0,\ldots,m)$ which reorders the components of $(n_0,n_1,\ldots,n_m)$ decreasingly, $n_{\lambda(0)}\geq\cdots\geq n_{\lambda(m)}$, and an associated Nikishin system $(r_{1,1},\ldots,r_{1,m})=\mathcal{N}(\rho_1,\ldots,\rho_m)$ such that

$$\mathcal{A}_{\mathbf{n},0}=(q_{\mathbf{n},0}+\sum_{k=1}^{m}q_{\mathbf{n},k}\widehat{r}_{1,k})\widehat{s}_{1,\lambda(0)},\quad \deg q_{\mathbf{n},k}\leq n_{\lambda(k)}-1,\quad k=0,\ldots,m,$$

where $\mathcal{A}_{\mathbf{n},0}=a_{\mathbf{n},0}t_0+\sum_{j=1}^{m}a_{\mathbf{n},j}t_j\hat{s}_{1,j}$ and $\widehat{s}_{1,\lambda(0)}\equiv 1$ when $\lambda(0)=0$. The permutation may be taken so that for all $0\leq j<k\leq n$ with $n_j=n_k$ then also $\lambda(j)<\lambda(k)$. In this case, see formulas (31) in the proof of [32, Lemma 2.3], it follows that $q_{\mathbf{n},m}$ is either $a_{\mathbf{n},\bar{j}}t_{\bar{j}}$ or $-a_{\mathbf{n},\bar{j}}t_{\bar{j}}$.

Set

$$\mathcal{Q}_{\mathbf{n},j}:=q_{\mathbf{n},j}+\sum_{k=j+1}^{m}q_{\mathbf{n},k}\widehat{r}_{1,k},\quad j=0,\ldots,m-1,\quad \mathcal{Q}_{\mathbf{n},m}:=q_{\mathbf{n},m}.$$

Suppose that $\lambda(0)=0$ and thus $\widehat{s}_{1,\lambda(0)}\equiv 1$. Then, $N_{\mathbf{n}}=n_0=n_{\lambda(0)}$. Due to *ii')*, it follows that

$$\frac{\mathcal{Q}_{\mathbf{n},0}(z)}{w_{\mathbf{n}}(z)}=\mathcal{O}(1/z^{|\mathbf{n}|-n_{\lambda(0)}-D})\in\mathcal{H}(\mathbb{C}\setminus\Delta_1). \tag{4.6}$$

When $\lambda(0) \neq 0$ we have that $N_{\mathbf{n}} = n_{\lambda(0)} - 1$ and $\widehat{s}_{1,\lambda(0)} = \mathcal{O}(1/z)$. Therefore, from *ii')* we again have (4.6). Using (3.2), it follows that

$$\int x^\nu \mathcal{Q}_{\mathbf{n},1}(x) \frac{d\rho_1(x)}{w_{\mathbf{n}}(x)} = 0, \qquad \nu = 0, \ldots, |\mathbf{n}| - n_{\lambda(0)} - D - 2.$$

This implies that $\mathcal{Q}_{\mathbf{n},1}$ has at least $|\mathbf{n}| - n_{\lambda(0)} - D - 1$ sign changes in $\overset{\circ}{\Delta}_1$. Let $w_{\mathbf{n},1}$ be the monic polynomial whose zeros are the points where $\mathcal{Q}_{\mathbf{n},1}$ changes sign in $\overset{\circ}{\Delta}_1$. Then

$$\frac{\mathcal{Q}_{\mathbf{n},1}(z)}{w_{\mathbf{n},1}(z)} = \mathcal{O}\left(1/z^{|\mathbf{n}| - n_{\lambda(0)} - n_{\lambda(1)} - D}\right) \in \mathcal{H}(\mathbb{C} \setminus \Delta_2).$$

Using again (3.2) we get

$$\int x^\nu \mathcal{Q}_{\mathbf{n},2}(x) \frac{d\rho_2(x)}{w_{\mathbf{n},1}(x)} = 0, \quad \nu = 0, \ldots, |\mathbf{n}| - n_{\lambda(0)} - n_{\lambda(1)} - D - 2,$$

which implies that $\mathcal{Q}_{\mathbf{n},2}$ has at least $|\mathbf{n}| - n_{\lambda(0)} - n_{\lambda(1)} - D - 1$ sign changes on $\overset{\circ}{\Delta}_2$. Repeating the arguments m times, it follows that $\mathcal{Q}_{\mathbf{n},m} = q_{\mathbf{n},m}$ has at least $n_{\overline{\jmath}} - D - 1$ sign changes on $\overset{\circ}{\Delta}_m$ which implies the statement because $q_{\mathbf{n},m} = \pm a_{\mathbf{n},\overline{\jmath}} t_{\overline{\jmath}}$ and the zeros of $t_{\overline{\jmath}}$ are outside Δ_m.

By Lemma 3.2.3, for $j = 0, \ldots m - 1$, we have

$$h - \lim_{\mathbf{n} \in \Lambda} \frac{a_{\mathbf{n},j}}{a_{\mathbf{n},m}} = (-1)^{m-j} \frac{t_m \widehat{s}_{m,j+1}}{t_j}, \qquad \text{inside} \qquad \mathbb{C} \setminus \Delta_m. \tag{4.7}$$

From (4.7) and [37, Lemma 1] it follows that each zero ζ of t_j of multiplicity κ attracts at least κ zeros of $a_{\mathbf{n},m}$ when $|\mathbf{n}| \to \infty$, $\mathbf{n} \in \Lambda$ (recall that t_m and t_j are relatively prime). Let us show that the number of zeros attracted by ζ equals κ and the rest of the zeros of $a_{\mathbf{n},m}$ either lie in $\overset{\circ}{\Delta}_m$ or accumulate on $\Delta_m \cup \{\infty\}$. Then [37, Lemma 1] and (4.7) imply (4.4).

The index $\overline{\jmath}$ as defined above may depend on $\mathbf{n} \in \Lambda$. Given $\overline{\jmath} \in \{0, \ldots, m\}$, let $\Lambda(\overline{\jmath})$ denote the set of all $\mathbf{n} \in \Lambda$ such that $\overline{\jmath}$ is the last component of $(n_0, \ldots, n_m)$ satisfying $n_{\overline{\jmath}} = \min_{j=0,\ldots,m}(n_j)$. Fix $\overline{\jmath}$ and suppose that $\Lambda(\overline{\jmath})$ contains infinitely many multi-indices. Should $\overline{\jmath} = m$, then $a_{\mathbf{n},m}$ has $n_m - D - 1$ zeros in $\overset{\circ}{\Delta}_m$ and the rest of its zeros converge to the zeros of the $t_j, j = 0, \ldots, m-1$, according to their multiplicity as needed.

Now, consider that $\overline{\jmath} \neq m$. By Lemma 3.2.3 we have

$$h - \lim_{\mathbf{n} \in \Lambda(\overline{\jmath})} \frac{a_{\mathbf{n},m}}{a_{\mathbf{n},\overline{\jmath}}} = (-1)^{m-\overline{\jmath}} \frac{t_{\overline{\jmath}}}{t_m \widehat{s}_{m,\overline{\jmath}+1}} \qquad \text{inside} \qquad \mathbb{C} \setminus \Delta_m. \tag{4.8}$$

For each $j \in \{0, \ldots, m-1\} \setminus \{\overline{\jmath}\}$ and each zero ζ of multiplicity κ of t_j choose κ zeros of $a_{\mathbf{n},m}$ that converge to ζ as $|\mathbf{n}| \to \infty, \mathbf{n} \in \Lambda(\overline{\jmath})$. Let q_n be a monic polynomial with this set of points as its zeros. Obvioustly, $\lim_{\mathbf{n} \in \Lambda(\overline{\jmath})} q_{\mathbf{n}} = \prod_{k=0}^{m-1} t_k / t_{\overline{\jmath}}$ (uniformly on compact subsets). From (4.8), we get

$$h - \lim_{n \in \Lambda(\overline{\jmath})} \frac{a_{\mathbf{n},m}}{q_{\mathbf{n}} a_{\mathbf{n},\overline{\jmath}}} = (-1)^{m-\overline{\jmath}} \frac{t_{\overline{\jmath}}^2}{\prod_{k=0}^{m} t_k \widehat{s}_{m,\overline{\jmath}+1}} \qquad \text{inside} \qquad \mathbb{C} \setminus \Delta_m. \tag{4.9}$$

Applying once more [37, Lemma 1], it follows that for $j \in \{0, \ldots, m\} \setminus \{\overline{\jmath}\}$ each zero of t_j attracts as many zeros of $a_{\mathbf{n},\overline{\jmath}}$ as its multiplicity (notice that the zeros of $q_{\mathbf{n}}$ were cancelled by zeros of $a_{\mathbf{n},m}$). Therefore, all the zeros of $a_{\mathbf{n},\overline{\jmath}}$ are located either on Δ_m or on a sufficiently small neighborhood of the zeros of the polynomial $\prod_{k=0}^{m} t_k / t_{\overline{\jmath}}$. Consequently,

$$\lim_{\mathbf{n} \in \Lambda(\overline{\jmath})} \frac{a_{\mathbf{n},m}}{a_{\mathbf{n},\overline{\jmath}}} = (-1)^{m-\overline{\jmath}} \frac{t_{\overline{\jmath}}}{t_m \widehat{s}_{m,\overline{\jmath}+1}} \qquad \text{inside} \qquad (\mathbb{C} \setminus \Delta_m)'. \tag{4.10}$$

The function on the right hand of (4.10) is meromorphic in $\mathbb{C} \setminus \Delta_m$. Its zeros correspond with those of $t_{\overline{\jmath}}$ (multiplicity included) and its poles are the zeros of t_m with order equal to the multiplicity of the zero. Using the argument principle, from (4.10) it follows that for each $j \in \{0, \ldots, m-1\}$ if ζ is a zero of t_j of multiplicity κ then ζ attracts exactly κ zeros of $a_{\mathbf{n},m}$ as $|\mathbf{n}| \to \infty, \mathbf{n} \in \Lambda(\overline{\jmath})$, and the remaining zeros of $a_{\mathbf{n},m}$ accumulate of Δ_m. This is true for each $\overline{\jmath}$. Hence the statement about the zeros of $a_{\mathbf{n},m}$ is valid for $\mathbf{n} \in \Lambda$ and (4.4) is satisfied.

Combining (4.4), the knowledge we have about the asymptotic behavior of the zeros of $a_{\mathbf{n},m}$, and the argument principle, we obtain the statement about the asymptotic behavior of the zeros of the $a_{\mathbf{n},j}, j = 0, \ldots, m-1$.

Now,

$$\frac{\mathcal{A}_{\mathbf{n},j}}{a_{\mathbf{n},m}} = \frac{a_{\mathbf{n},j} t_j}{a_{\mathbf{n},m}} + \sum_{k=j+1}^{m-1} \frac{a_{\mathbf{n},k} t_k}{a_{\mathbf{n},m}} \widehat{s}_{j+1,k} + \widehat{s}_{j+1,m}.$$

According to formula (17) in [33, Lemma 2.9]

$$0 \equiv (-1)^{m-j} \widehat{s}_{m,j+1} + \sum_{k=j+1}^{m-1} (-1)^{m-k} \widehat{s}_{m,k+1} \widehat{s}_{j+1,k} + \widehat{s}_{j+1,m}, \quad z \in \mathbb{C} \setminus (\Delta_{j+1} \cup \Delta_m).$$

Multiply both sides of this equation by t_m and delete the resulting expression from the previous one. We have that

$$\frac{\mathcal{A}_{\mathbf{n},j}}{a_{\mathbf{n},m}} = t_j\left(\frac{a_{\mathbf{n},j}}{a_{\mathbf{n},m}} - (-1)^{m-j}\frac{t_j}{t_m}\widehat{s}_{m,j+1}\right) + \sum_{k=j+1}^{m-1} t_k\left(\frac{a_{\mathbf{n},k}}{a_{\mathbf{n},m}} - (-1)^{m-k}\frac{t_j}{t_m}\widehat{s}_{m,k+1}\right) \tag{4.11}$$

Consequently, for each $j = 0, \ldots, m-1$, from (4.4) we obtain

$$\lim_{\mathbf{n}\in\Lambda} \frac{\mathcal{A}_{\mathbf{n},j}}{a_{\mathbf{n},m}} = 0$$

uniformly on each compact subset of $\mathbb{C} \setminus (\Delta_{j+1} \cup \Delta_m)'$ which is (4.5). $\square$

An important consequence of this last result is the following theorem

Theorem 4.1.2. *Let* $(s_{1,1}, \ldots, s_{1,m}) = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$ *and* $\Lambda \subset \mathbb{Z}_+^{m+1}$ *be given which verifies* (3.14). *Let* $(t_0, \ldots, t_m)$ *be a vector polynomial with real coefficients whose zeros lie in* $\mathbb{C} \setminus \Delta_m$ *and for* $j \neq k$ *the zeros of* t_j *and* t_k *are distinct. Assume that either* Δ_{m-1} *is bounded away from* Δ_m *or* σ_m *satisfies* (4.1). *Then, for all* $\mathbf{n} \in \Lambda$ *with* $n_0 + \cdots + n_m$ *sufficiently large any linear form*

$$p_{\mathbf{n},0}t_0 + \sum_{j=1}^{m} p_{\mathbf{n},j}t_j\hat{s}_{1,j},$$

where $p_{\mathbf{n},0}, \ldots, p_{\mathbf{n},m}$ *are arbitrary polynomials with* $\deg p_{\mathbf{n},j} \leq n_j - 1$, *has at most* $|\mathbf{n}| - 1$ *zeros on* $\Delta \subset \mathbb{R}\setminus\Delta_1$. *In particular,* $(t_0, t_1\hat{s}_{1,1}, \ldots, t_m\hat{s}_{1,m})$ *is an AT system on any interval* $\Delta \subset \mathbb{R} \setminus \Delta_1$ *for the described set of multi-indices.*

Proof. There is no loss of generality if we consider multi-indices of the form $(n_0 - d_0, \ldots, n_m - d_m) \in \mathbb{Z}_+^{m+1}$ where $d_j = \deg t_j, j = 0, \ldots, m$. We will reason by contradiction.

Let us assume that their exists an infinite sequence of multi-indices $\Lambda' \subset \Lambda$ such that for each $\mathbf{n} \in \Lambda'$ their exist polynomials $p_{\mathbf{n},0}, \ldots, p_{\mathbf{n},m}, \deg p_{\mathbf{n},j} \leq n_j - d_j - 1$ with real coefficients, not all identically equal to zero, for which $p_{\mathbf{n},0}t_0 + \sum_{j=1}^{m} p_{\mathbf{n},j}t_j\hat{s}_{1,j}$ has at least $|\mathbf{n}| - D$ zeros on $\Delta \subset \mathbb{R} \setminus \Delta_1$, where $D = \sum_{j=0}^{m} d_j$. Let $w_{\mathbf{n}}$ be the polynomial whose zeros are those of $p_{\mathbf{n},0}t_0 + \sum_{j=1}^{m} p_{\mathbf{n},j}t_j\hat{s}_{1,j}$ on $\mathbb{C} \setminus \Delta_1$. Then, it is easy to check that

$$(p_{\mathbf{n},0}t_0 + \sum_{j=1}^{m} p_{\mathbf{n},j}t_j\hat{s}_{1,j})/w_{\mathbf{n}} = \mathcal{O}(1/z^{|\mathbf{n}|-N_{\mathbf{n}}-D+1}) \in \mathcal{H}(\mathbb{C} \setminus \Delta_1).$$

Therefore, the polynomials $p_{\mathbf{n},0},\ldots,p_{\mathbf{n},m}$ fulfill *i')-ii')*, with an extra power of $1/z$ in the right hand of *ii')*.

Let $\overline{\jmath}$ and $\Lambda(\overline{\jmath})$ be defined as in the proof of Theorem 4.1.1. Obviously, $\Lambda(\overline{\jmath})$ must contain infinitely many multi-indices in Λ' for some $\overline{\jmath}\in\{0,\ldots,m\}$. Fix $\overline{\jmath}$ so that this occurs. Arguing as in the proof of Theorem 4.1.1, we obtain that $p_{\mathbf{n},\overline{\jmath}}t_{\overline{\jmath}}$ has at least $n_{\overline{\jmath}}-D$ zeros on Δ_m and for all sufficiently large $|\mathbf{n}|$ as many zeros close to each one of the zeros of $t_j, j=0,\ldots,m$, as their multiplicity. Therefore, for all sufficiently large $|\mathbf{n}|, \mathbf{n}\in\Lambda(\overline{\jmath})\cap\Lambda'$ we have that $\deg p_{\mathbf{n},\overline{\jmath}}t_{\overline{\jmath}}=n_{\overline{\jmath}}$. This contradicts the fact that by construction $\deg p_{\mathbf{n},\overline{\jmath}}t_{\overline{\jmath}}\le n_{\overline{\jmath}}-1$. Thus our initial assumption is false and the statement of the theorem true. □

4.2 Type II Hermite-Padé approximants (real case)

In all that follows, when we write $\mathcal{O}(1/z^N)$ it is understood that $z\to\infty$ and the limit is taken along any curve which is not tangent to the half straight line containing the support of the measures under consideration. By T we denote the least common multiple of the denominators $t_1,\ldots,t_m$ of the rational functions $r_1,\ldots,r_m$. In this section $\mathbf{n}=(n_1,\ldots,n_m)\in\mathbb{Z}_+^m\setminus\{\mathbf{0}\}$ and $|\mathbf{n}|:=n_1+\cdots+n_m$ and the vector of rational fractions $\mathbf{r}$ have real coefficients.

Lemma 4.2.1. *Let* $\mathbf{R}_n=(P_{\mathbf{n},1}/Q_{\mathbf{n}},\ldots,P_{\mathbf{n},m}/Q_{\mathbf{n}})$ *be a type II Hermite-Padé approximant with respect to* $\mathbf{f}=\mathbf{s}+\mathbf{r}$ *and* $\mathbf{n}\in\mathbb{Z}_+^m\setminus\{\mathbf{0}\}$ *. Assume that* $n_j>D:=\deg T$, $j=1\ldots,m$*. Then, for each* $j=1,\ldots,m$

$$\int x^\nu t_j(x)Q_{\mathbf{n}}(x)ds_{1,j}(x)=0,\qquad \nu=0,1,\ldots n_j-d_j-1,\tag{4.12}$$

where $d_j=\deg t_j$*. It follows that for any polynomials* p_j $\deg p_j\le n_j-d_j-1$, $j=1,\ldots,m$

$$\int Q_{\mathbf{n}}(x)\left(p_1t_1+\sum_{j=2}^m p_jt_j\widehat{s}_{2,j}\right)(x)d\sigma_1(x)=0;\tag{4.13}$$

and for any polynomials $p_j,\deg p_j\le n_j-D-1, j=1,\ldots,m$

$$\int (TQ_{\mathbf{n}})(x)\left(p_1+\sum_{j=2}^m p_j\widehat{s}_{2,j}\right)(x)d\sigma_1(x)=0.\tag{4.14}$$

Hence $Q_{\mathbf{n}}$ has at least $|\mathbf{n}| - mD$ zeros in $\mathring{\Delta}_1$. Moreover,

$$(TQ_{\mathbf{n}}\widehat{s}_{1,j} + TQ_{\mathbf{n}}(v_j/t_j) - TP_{\mathbf{n},j})(z) = \int \frac{(TQ_{\mathbf{n}})(x)}{z-x} ds_{1,j}(x). \tag{4.15}$$

Proof. Let $(P_{\mathbf{n},1}/Q_{\mathbf{n}}, \ldots, P_{\mathbf{n},m}/Q_{\mathbf{n}})$ be a type II Hermite-Padé approximant with respect to $\mathbf{f}$ then for each $j = 1, \ldots, m$,

$$Q_{\mathbf{n}}(z)\left(\hat{s}_j + \frac{v_j}{t_j}\right)(z) - P_{\mathbf{n},j}(z) = \mathcal{O}\left(\frac{1}{z^{n_j+1}}\right) \in \mathcal{H}\left(\mathbb{C} \setminus (\Delta \cup \aleph_j)\right), \tag{4.16}$$

where $\aleph_j$ is the set of zeros of t_j. Take $\mathbf{n}$ so that $n_j \geq D, j = 1, \ldots, m$. For each j we multiplying (4.16) by t_j then we obtain

$$\left(Q_{\mathbf{n}} t_j \hat{s}_j + Q_{\mathbf{n}} v_j - t_j P_{\mathbf{n},j}\right)(z) = \mathcal{O}\left(\frac{1}{z^{n_j-d_j+1}}\right) \in \mathcal{H}\left(\mathbb{C} \setminus \Delta\right),$$

and using (3.2) in Lemma 3.1.2, relation (4.12) follows. Taking linear combinations of the relations given by (4.12) we arrive at (4.13).

In (4.12) we can replace $x^\nu, \nu = 0, \ldots, n_j - d_j - 1$, by $x^\nu T/t_j, \nu = 0, \ldots, n_j - D - 1$ and taking linear combinations of the orthogonality relations thus obtained we get (4.14).

From [33, Theorem 1.1] we know that $(1, \hat{s}_{2,2}, \ldots, \hat{s}_{2,m})$ forms an AT-system on Δ_1. Suppose that $Q_{\mathbf{n}}$ has at most $|\mathbf{n}| - mD - 1$ sign changes in $\mathring{\Delta}_1$. Then, we can choose $p_1 \ldots, p_m$ conveniently so that $p_1 + \sum_{j=2}^m p_j \widehat{s}_{2,j}, \deg p_j \leq n_j - D - 1, j = 1, \ldots, m$, has simple zeros at the points of sign change of $Q_{\mathbf{n}}$ on $\mathring{\Delta}_1$ and no other zero in Δ_1. This contradicts (4.14). Thus, $Q_{\mathbf{n}}$ has at least $|\mathbf{n}| - mD$ zeros in $\mathring{\Delta}_1$. Finally, (4.15) is a consequence of (3.1) in Lemma 3.1.2. □

In Section 3.1 we advanced some facts about Nikishin systems, now we need to introduced additional properties that we will summarize below.

As we mentioned in Section 3.1 given a measure $\sigma \in \mathcal{M}(\Delta)$, where Δ is contained in a half line, there exists a measure $\tau \in \mathcal{M}(\Delta)$ and $\ell(z) = az + b, a = 1/|\sigma|, b \in \mathbb{R}$, such that

$$1/\widehat{\sigma}(z) = \ell(z) + \widehat{\tau}(z), \tag{4.17}$$

where $|\sigma|$ is the total variation of the measure σ. Recall that given a measure $s_{j,k}$ by $\tau_{j,k}$ we denote its inverse and $\ell_{j,k}$ the corresponding polynomial in (4.17). That is,

$$1/\widehat{s}_{j,k}(z) = \ell_{j,k}(z) + \widehat{\tau}_{j,k}(z).$$

For each $j \in \{2,\ldots,m\}$ we define an auxiliary Nikishin system

$$S^j = (s^j_{2,2},\ldots,s^j_{2,m}) = \mathcal{N}(\sigma^j_2,\ldots,\sigma^j_m) :=$$

$$\mathcal{N}(\tau_{2,j},\widehat{s}_{2,j}d\tau_{3,j},\ldots,\widehat{s}_{j-1,j}d\tau_{j,j},\widehat{s}_{j,j}d\sigma_{j+1},\sigma_{j+2},\ldots,\sigma_m)\,.$$

We also define $S^1 = (s^1_{2,2},\ldots,s^1_{2,m}) = \mathcal{N}(\sigma_2,\ldots,\sigma_m)$.

These auxiliary systems were used in the proof of [15, Lemmas 5-6] (see also [28, Theorem 3.1.3]). Subsequently, in [32, Lemma 3.2], several formulas involving ratios of Cauchy transforms were proved (when the supports of the generating measures are unbounded or have a common end point see [33, Lemma 2.10] instead). For convenience of the reader, we write some of those formulas which we will employ using the notation introduced above. We have:

$$\frac{1}{\widehat{s}_{2,j}} = \ell_{2,j} + \widehat{\tau}_{2,j}\,, \tag{4.18}$$

$$\frac{\widehat{s}_{1,k}}{\widehat{s}_{1,1}} = \frac{|s_{1,k}|}{|s_{1,1}|} - \widehat{\langle \tau_{1,1},\langle s_{2,k},\sigma_1\rangle\rangle}, \qquad 1 < k \le m, \tag{4.19}$$

$$\frac{\widehat{s}_{2,k}}{\widehat{s}_{2,j}} = a_{j,k} + (-1)^{k-1}\widehat{s}^j_{2,k+1} + c_{j,k}\widehat{s}^j_{2,k}\,, \qquad k = 2,\ldots,j-1\,, \tag{4.20}$$

and

$$\frac{\widehat{s}_{2,k}}{\widehat{s}_{2,j}} = a_{j,k} + c_{j,k}\widehat{s}^j_{2,k}(z)\,, \qquad k = j+1,\ldots,m\,, \tag{4.21}$$

where the $a_{j,k}$ and $c_{j,k}$ denote (perfectly determined) constants.

Definition 4.2.2. *Let* $\mathbf{n} = (n_1,\ldots,n_m) \in \mathbb{Z}^m_+$. *For each* $j = 1,\ldots,m$, *we define an associated multi-index* $\mathbf{n}^j = (n^j_2,\ldots,n^j_m)$ *whose* $m-1$ *components are given by*

$$n^j_k = \begin{cases} \min(n_1,\ldots,n_{k-1},n_j-1), & \textit{when} \quad k = 2,\ldots,j, \\ \\ \min(n_j,n_k), & \textit{when} \quad k = j+1,\ldots,m\,. \end{cases}$$

We denote $|\mathbf{n}^j| = \sum_{k=2}^m n^j_k$.

For $j = 1,,\ldots,m$, set $\Phi_{\mathbf{n},j} := (TQ_{\mathbf{n}}\widehat{s}_{1,j} + TQ_{\mathbf{n}}(v_j/t_j) - TP_{\mathbf{n},j})$ (see (4.15)).

Lemma 4.2.3. *Let* $\mathbf{n} = (n_1, \ldots, n_m)$ *be a multi-index such that* $n_k > D = \deg T$, $k = 1, \ldots, m$. *Then, for each* $j = 1, \ldots m$

$$\int (p_k \Phi_{\mathbf{n},j})(x) ds^j_{2,k}(x) = 0, \quad k = 2, \ldots, m, \tag{4.22}$$

where p_k *are arbitrary polynomials with* $\deg p_k < n^j_k - D$.

Set $N^j = |\mathbf{n}^j| + n_j$. *For each* $j = 1, \ldots, m$, *there exists a monic polynomial* $w_{\mathbf{n},j}$, $\deg w_{\mathbf{n},j} \geq |\mathbf{n}^j| - (m-1)D$, *with simple zeros which lie in* $\overset{\circ}{\Delta}_2$, *such that*

$$\frac{\Phi_{\mathbf{n},j}(z)}{w_{\mathbf{n},j}(z)} = \left[\frac{(TQ_{\mathbf{n}}(\hat{s}_{1,j} + r_j) + P_{n,j})(z)}{w_{\mathbf{n},j}(z)} \right] = \mathcal{O}\left(\frac{1}{z^{N^j - mD + 1}} \right) \in H(\mathbb{C} \setminus \Delta_1). \tag{4.23}$$

and

$$0 = \int x^\nu T(x) Q_{\mathbf{n}}(x) \frac{ds_{1,j}(x)}{w_{\mathbf{n},j}(x)}, \qquad \nu = 0, 1, \ldots N^j - mD - 1. \tag{4.24}$$

Proof. Fix $j \in \{1, \ldots, m\}$. From the definition of $\Phi_{\mathbf{n},j}$ and (4.15), we obtain

$$\int (p_k \Phi_{\mathbf{n},j})(x) ds^j_{2,k}(x) = \int p_k(x) \int \frac{T(t) Q_{\mathbf{n}}(t)}{x - t} ds_{1,j}(t) ds^j_{2,k}(x).$$

Since $\deg p_k < n^j_k - D \leq n_j - D$, from (4.12) and Fubini's theorem, it follows that

$$\int p_k(x) \int \frac{T(t) Q_{\mathbf{n}}(t)}{x - t} ds_{1,j}(t) ds^j_{2,k}(x) =$$

$$\int \int \frac{(p_k T Q_{\mathbf{n}})(t)}{x - t} ds_{1,j}(t) ds^j_{2,k}(x) = - \int (p_k T Q_{\mathbf{n}} \widehat{s}^j_{2,k})(t) ds_{1,j}(t). \tag{4.25}$$

First, we prove the statement of the lemma for the case $j + 1 \leq k \leq m$. If $j = m$, the set of such values of k is empty and there is nothing to prove. Let $j \leq m - 1$. Using (4.21), we obtain

$$- \int (p_k T Q_{\mathbf{n}} \widehat{s}^j_{2,k})(t) ds_{1,j}(t) = \int (p_k T Q_{\mathbf{n}})(t) \left(\frac{\widehat{s}_{2,k}(t)}{\widehat{s}_{2,j}(t)} - a_{j,k} \right) ds_{1,j}(t) =$$

$$-a_{j,k} \int (p_k T Q_{\mathbf{n}})(t) ds_{1,j}(t) + \int (p_k T Q_{\mathbf{n}})(t) ds_{1,k}(t).$$

In the last equality we have the sum of two terms. By hypothesis $\deg p_k < n^j_k - D \leq \min\{n_j - D, n_k - D\}$. Taking (4.12) into account, we deduce that both terms vanish. Hence the first case is proved.

Now, we analyze the case when $2 \leq k \leq j$. Using several times formula (4.20) to make k descend to 2 and finally formula (4.18), we obtain the equalities

$$\widehat{s}^{\,j}_{2,k} = (-1)^k \left(\frac{\widehat{s}_{2,k-1}}{\widehat{s}_{2,j}} - a_{j,k-1} - c_{j,k-1}\widehat{s}^{\,j}_{2,k-1} \right) = \tag{4.26}$$

$$(-1)^k \left(\frac{\widehat{s}_{2,k-1}}{\widehat{s}_{2,j}} - a_{j,k-1} - (-1)^{k-1} c_{j,k-1} \left(\frac{\widehat{s}_{2,k-2}}{\widehat{s}_{2,j}} - a_{j,k-2} - c_{j,k-2}\widehat{s}^{\,j}_{2,k-2} \right) \right) =$$

$$\cdots = \mathcal{L}^*_j + \frac{1}{\widehat{s}_{2,j}} \sum_{l=1}^{k-1} c^*_l \widehat{s}_{2,l} \, ,$$

where $\mathcal{L}^*_j$ denotes a polynomial of degree 1, $\widehat{s}_{2,1} \equiv 1$, and the $c^*_l, l = 1, \ldots, k-1$, are constants. Sustituting (4.26) into (4.25), we obtain

$$\int (p_k T Q_{\mathbf{n}} \widehat{s}^{\,j}_{2,k})(t) ds_{1,j}(t) =$$

$$-\int (p_k T Q_{\mathbf{n}})(t) \mathcal{L}^*_j(t) ds_{1,j}(t) - \sum_{l=1}^{k-1} c^*_i \int (p_k T Q_{\mathbf{n}})(t) ds_{1,l}(t) \, .$$

From hypothesis $\deg p_k \leq \min(n_1 - D - 1, \ldots, n_{k-1} - D - 1, n_j - D - 2)$ and using (4.12) it follows that all the integrals on the right hand side of this equality are zero. Hence, (4.22) holds.

From (4.22) it follows that for any polynomials $p_k, \deg p_k < n^j_k - D, k = 2, \ldots, m$

$$\int \Phi_{\mathbf{n},j}(x)(p_2 + \sum_{k=3}^{m} p_k \widehat{s}^{\,j}_{3,k})(x) d\sigma^j_2(x) = 0. \tag{4.27}$$

According to [33, Theorem 1.1], $(1, \widehat{s}^{\,j}_{3,3}, \ldots, \widehat{s}^{\,j}_{3,m})$ forms an AT system on Δ_2. Thus, using (4.27) it follows that $\Phi_{\mathbf{n},j}$ has at least $|\mathbf{n}^j| - (m-1)D$ changes of sign in $\mathring{\Delta}_2$. Let $w_{\mathbf{n},j}$ be the monic polynomial whose zeros are the points where $\Phi_{\mathbf{n},j}$ changes of sign in the interior of Δ_2. Taking into account that $\deg w_{\mathbf{n},j} \geq |\mathbf{n}^j| - (m-1)D$, we obtain (4.23) which together with (3.2) implies (4.24). $\square$

We are ready to prove convergence of type II Hermite Padé approximants in Hausdorff content.

Lemma 4.2.4. *Let* $(s_{1,1}, \ldots, s_{1,m}) = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$ *and* $(r_1, \ldots, r_m)$ *be given, where the rational functions* $r_j, j = 1, \ldots, m$ *have real coefficients and their poles lie in* $\mathbb{C} \setminus \Delta_1$. *Let* $\Lambda \subset \mathbb{Z}_+^m$ *be an infinite sequence of distinct multi-indices satisfying* (3.14). *Assume that either* Δ_2 *is bounded away from* Δ_1 *or* σ_1 *satisfies* (4.1). *Then, for* $j = 1, \ldots, m$

$$h - \lim_{\mathbf{n} \in \Lambda} \frac{P_{\mathbf{n},j}}{Q_{\mathbf{n}}} = f_j = \widehat{s}_{1,j} + r_j, \qquad \textit{inside} \qquad \mathbb{C} \setminus \Delta_1.$$

Proof. Fix $j = 1, \ldots, m$. According to Lemma 4.2.3, the zeros of $w_{\mathbf{n},j}$ lie in $\mathbb{R} \setminus \Delta_1$. Using (4.23) and (3.14), it is easy to verify that for all sufficiently large $|\mathbf{n}|$ the rational function $P_{\mathbf{n},j}/Q_{\mathbf{n}}$ is an incomplete Padé approximant of f_j taking $n = |\mathbf{n}|, \kappa = -\mathbf{1}$ and choosing ℓ appropriately. From Theorem 3.1.5 we have that if σ_1 satisfies Carleman's condition so does $s_{1,j} = \langle \sigma_1, \langle \sigma_2, \ldots, \sigma_j \rangle \rangle$. Therefore, the statement readily follows from Lemma 3.2.2. □

Now, we can state and proof the main result of this chapter.

Theorem 4.2.5. *Let* $\mathbf{s} = (s_{1,1}, \ldots, s_{1,m}) = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$ *and* $\mathbf{r} = (r_1, \ldots, r_m)$ *be given, where the rational functions* $r_j, j = 1, \ldots, m$ *have real coefficients, for different* j *their poles are distinct, and they all lie in* $\mathbb{C} \setminus (\Delta_1 \cup \Delta_m)$. *Let* $\Lambda \subset \mathbb{Z}_+^m$ *be an infinite sequence of distinct multi-indices which verifies* (3.14). *Assume that* Δ_2 *is bounded away from* Δ_1 *or* σ_1 *satisfies* (4.1), *and* Δ_{m-1} *is bounded away from* Δ_m *or* σ_m *satisfies* (4.1). *Let* $(\mathbf{R_n})_{\mathbf{n} \in \Lambda}$ *be the corresponding sequence of type II Hermite-Padé approximants of* $\mathbf{f} = \mathbf{s} + \mathbf{r}$. *Then, for* $j = 1, \ldots m$

$$\lim_{\mathbf{n} \in \Lambda} \frac{P_{\mathbf{n},j}}{Q_{\mathbf{n}}} = f_j = \widehat{s}_{1,j} + r_j, \qquad \textit{inside} \qquad (\mathbb{C} \setminus \Delta_1)', \tag{4.28}$$

the set obtained deleting from $\mathbb{C} \setminus \Delta_1$ *the poles of all the* r_j. *For each* $\varepsilon > 0$ *sufficiently small, there exists* $N > 0$ *such that for all* $\mathbf{n} \in \Lambda$ *with* $\sum_{k=1}^m n_k \geq N$ *we have* $\deg Q_{\mathbf{n}} = \sum_{k=1}^m n_k$, *if* ζ *is a pole of some* r_j *of order* κ *then* $Q_{\mathbf{n}}$ *has exactly* κ *zeros in the disk* $\{z : |z - \zeta| < \varepsilon\}$, *and* $Q_{\mathbf{n}}$ *has exactly* $\sum_{k=1}^m (n_k - d_k)$ *simple zeros in* $\overset{\circ}{\Delta}_1$ *(the interior of* Δ_1 *with the Euclidean topology of* $\mathbb{R}$*).*

Proof. By Theorem 4.1.2 we know that there exists an N such that $(t_1, t_2\hat{s}_{2,2}, \ldots, t_m\hat{s}_{2,m})$ forms an AT-system on Δ_1 for the multi-index $(n_1 - d_1, n_2 - d_2, \ldots, n_m - d_m)$ if $|\mathbf{n}| > N$. From (4.13) it follows that $Q_{\mathbf{n}}$ has at least $|\mathbf{n}| - D, D := d_1 + d_2 + \ldots + d_m$ zeros in $\overset{\circ}{\Delta}_1$ if $|\mathbf{n}| > N$.

Let ζ be a pole of r_j of order κ. Using Lemma 4.2.4 and [37, Lemma 1]) it follows that for any $\varepsilon > 0$ sufficiently small $Q_{\mathbf{n}}$ has at least κ zeros in $\{z : |z-\zeta| < \varepsilon\}$. The poles of the different r_j are distinct. Thus, for each $\varepsilon > 0$ and all sufficiently large $|\mathbf{n}|$ the zeros of $Q_{\mathbf{n}}$ are either in $\overset{\circ}{\Delta}_1$ or inside an ε neighborhood of the poles of the r_j. Lemma 4.2.4 and [37, Lemma 1]) then imply (4.28). $\square$

Remark 4.2.6. *Let*

$$r_j(z) = \sum_{l=1}^{L^j} \sum_{k=1}^{M_l^j} \frac{A_{l,k}^j}{(k-1)!} \cdot \frac{1}{(z-a_l^j)^k}$$

where $a_1^j, \ldots, a_{L^j}^j$ are the poles of the rational fraction r_j, $A_{l,M_l^j}^j \neq 0$, and M_l^j is the multiplicity of the pole a_l^j.

It is easy to check that for all $j = 1, \ldots, m$, $Q_{\mathbf{n}}$ satifies

$$0 = \int x^\nu Q_{\mathbf{n}}(x) ds_{1,j}(x) + \sum_{l=1}^{L^j} \sum_{k=1}^{M_l^j} A_{l,k}^j (x^\nu Q_{\mathbf{n}}(x))^{(k-1)} \mid_{x=a_l^j} \qquad \nu = 0, \ldots n_j - 1 \quad (4.29)$$

In other words $Q_{\mathbf{n}}$ is the type II multiple orthogonal polynomials respect to $(\mathbf{S}, |\mathbf{n}|+1, \mathbf{n})$ where $\mathbf{S}$ is the system

$$\mathbf{S} = \begin{pmatrix} s_{1,1} + \sum_{l=1}^{L^1} \sum_{k=1}^{M_l^1} A_{l,k}^1 \delta_{a_l^1}^{(k-1)} \\ \vdots \\ s_{1,m} + \sum_{l=1}^{L^m} \sum_{k=1}^{M_l^m} A_{l,k}^m \delta_{a_l^m}^{(k-1)} \end{pmatrix}, \qquad (4.30)$$

where $\int f(x) d\delta_a^{(k-1)}(x) = f^{(k-1)}(a)$.

Consider the type I multiple orthogonal polynomials $(a_{\mathbf{n},1} \ldots, a_{\mathbf{n},m})$ with respect to $(\mathbf{S}^T, \mathbf{n}, |\mathbf{n}| - 1)$. Then from Theorem 4.2.5 and Proposition 2.1.5 it follows that for all $\mathbf{n} \in \Lambda$, with

$$\sup_{\mathbf{n}\in\Lambda} \left(\max_{j=1,\ldots,m} (n_j) - \min_{k=1,\ldots,m} (n_k) \right) \leq C < \infty,$$

and sufficiently large $|\mathbf{n}|$, $\deg a_{\mathbf{n},j} = n_j - 1, j = 1, \ldots, m$. That is $(\mathbf{n}, |\mathbf{n}| - 1)$ is normal with respect to the matrix $\mathbf{S}^T$.

4.3 Type II Hermite-Padé approximants (complex case)

In this section we analyze the convergence of type II Hermite-Padé approximants with respect to the system of functions

$$f_j(z) = \widehat{s}_{1,j}(z) + r_j(z), \qquad j = 1, \ldots, m. \tag{4.31}$$

where the rational fractions r_j have comlpex coefficents. In doing so, we have to deal with the ratio asymptotic behaviour of type II multiple orthogonal polynomials with respect to a Nikishin system of measures (see [8] and [49]). Such results extend Rakhmanov's theorem on the ratio asymptotics of orthogonal polynomials (see [66, 67, 68]).

We need to introduce some concepts in order to formulate the result on ratio asymptotics that we will use.

Let $\Delta_1, \ldots, \Delta_m$ be bounded intervals of the real line. Consider the $(m+1)$-sheeted Riemann surface

$$\mathcal{R} = \overline{\bigcup_{k=0}^{m} \mathcal{R}_k},$$

formed by the consecutively "glued" sheets

$$\mathcal{R}_0 := \overline{\mathbb{C}} \setminus \Delta_1, \quad \mathcal{R}_k := \overline{\mathbb{C}} \setminus (\Delta_k \cup \Delta_{k+1}),\ k = 1, \ldots, m-1, \quad \mathcal{R}_m = \overline{\mathbb{C}} \setminus \Delta_m,$$

where the upper and lower banks of the slits of two neighboring sheets are identified. Fix $l \in \{1, \ldots, m\}$. There exists a conformal representation $\psi^{(l)}$ of $\mathcal{R}$ onto $\overline{\mathbb{C}}$ such that

$$\psi^{(l)}(z) = z + \mathcal{O}(1)\,,\ z \to \infty^{(0)}, \quad \psi^{(l)}(z) = C/z + \mathcal{O}(1/z^2)\,,\ z \to \infty^{(l)}.$$

By $\psi_k^{(l)}$ we denote the branch of $\psi^{(l)}$ on $\mathcal{R}_k$. In our case we are specially interested in $\psi_0^{(1)}$ which we will simply denote ψ. According to its construction we have that ψ defines a one-to-one analytic function on the region $\overline{\mathbb{C}} \setminus \Delta_1$ such that $\psi(\infty) = \infty$.

Given $n, k \in \mathbb{Z}_+ \setminus \{0\}$ define the multi-index

$$\mathbf{n}_k = (n+k, n, \ldots, n) \in \mathbb{Z}_+^m.$$

As a consequence of [49, Theorem 1.1]) we have the following result.

Lemma 4.3.1. *Let* $(s_{1,1}, \ldots, s_{1,m}) = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$ *be a Nikishin system such that for each* $j = 1, \ldots, m$ *the intervals* Δ_j *are bounded and* $|\sigma_j'| > 0$ *almost everywhere on* Δ_j. *Then*

$$\lim_{n \in \mathbb{Z}_+} \frac{Q_{\mathbf{n}_{k+1}}(z)}{Q_{\mathbf{n}_k}(z)} = \psi(z), \tag{4.32}$$

uniformly on each compact subset of $\mathbb{C} \setminus \Delta_1$.

From (4.32) it readily follows that for any fixed $0 \le j < k$ we have that

$$\lim_{n \in \mathbb{Z}_+} \frac{Q_{\mathbf{n}_k}(z)}{Q_{\mathbf{n}_j}(z)} = \psi^{k-j}(z), \tag{4.33}$$

uniformly on each compact subset of $\mathbb{C} \setminus \Delta_1$.

Let $(\frac{P_{\mathbf{n},1}}{Q_{\mathbf{n}}}, \ldots, \frac{P_{\mathbf{n},m}}{Q_{\mathbf{n}}})$ be the type II Hermite-Padé approximant with respect to the system (4.31) and $\mathbf{n} \in \mathbb{Z}_+^m \setminus \{\mathbf{0}\}$. As above we denote $r_j = v_j/t_j, \deg v_j < \deg t_j = d_j$ where v_j, t_j are mutually prime polynomials. As before $T, \deg T = D$, denotes the least common multiple of $t_1 \ldots, t_m$, with leading coefficient equal to one. From equation (4.12) we have that

$$\int x^\nu Q_{\mathbf{n}}(x) T(x) ds_{1,j}(x) = 0, \quad \nu = 0, 1, \ldots n_j - D - 1, \quad j = 1, \ldots, m \tag{4.34}$$

and by Lemma 3.1.2

$$(TQ_{\mathbf{n}} f_j - TP_{\mathbf{n},j})(z) = \int \frac{TQ_{\mathbf{n}}(x)}{z - x} ds_{1,j}(x). \tag{4.35}$$

Notice that these relations remain valid when the rational fractions $r_j = v_j/t_j$ have complex coefficients.

Fix $\mathbf{n} \in \mathbb{Z}_+^m \setminus \{\mathbf{0}\}$. Let $L_{\mathbf{n}}$ be the monic polynomial of degree $|\mathbf{n}|$ that satisfies the following multiple orthogonal relations

$$\int x^\nu L_{\mathbf{n}}(x) ds_{1,j}(x) = 0, \quad \nu = 0, 1, \ldots n_j - 1, \quad j = 1, \ldots m. \tag{4.36}$$

Our next objective is to express the polynomial $Q_{\mathbf{n}}$ in terms of the multiple orthogonal polynomials $L_{\mathbf{n}}$.

Given $\mathbf{n} = (n_1, \ldots, n_m)$ and $k \in \mathbb{Z}_+$ define

$$\overline{n} = \min\{n_k - D, k = 1, \ldots, m\}, \qquad \overline{\mathbf{n}}_k = (\overline{n} + k, \overline{n}, \ldots, \overline{n}) \in \mathbb{Z}_+^m.$$

Notice that, $\deg L_{\overline{\mathbf{n}}_k} = m\overline{n} + k$, all the zeros of $L_{\overline{\mathbf{n}}_k}$ are simple and lie on Δ_1. Moreover, from (4.36) we have that for each $k \geq 0$

$$\int x^\nu L_{\overline{\mathbf{n}}_k}(x) ds_{1,j}(x) = 0, \quad \nu = 0, \dots, \overline{n} - 1, \quad j = 1, \dots, m, \tag{4.37}$$

and (4.34) implies that

$$\int x^\nu Q_{\mathbf{n}}(x) T(x) ds_{1,j}(x) = 0, \quad \nu = 0, 1, \dots \overline{n} - 1, \quad j = 1, \dots, m \tag{4.38}$$

We have that $\deg TQ_{\mathbf{n}} \leq |\mathbf{n}| + D$, Set

$$N_{\mathbf{n}} = |\mathbf{n}| + D - m\overline{n}.$$

Lemma 4.3.2. *Consider the monic polynomial $TQ_{\mathbf{n}}$, then there exist unique constants $\lambda^*_{\mathbf{n},k}$, $k = 0, \dots, N_{\mathbf{n}}$, such that*

$$TQ_{\mathbf{n}} = \sum_{k=0}^{N_{\mathbf{n}}} \lambda^*_{\mathbf{n},k} L_{\overline{\mathbf{n}}_k} \,. \tag{4.39}$$

In particular, $\lambda^*_{\mathbf{n},N_{\mathbf{n}}} = 1$ if and only if $\deg TQ_{\mathbf{n}} = |\mathbf{n}| + D$.

Proof. Since $\deg TQ_{\mathbf{n}} \leq |\mathbf{n}| + D$, and $\{L_{\overline{\mathbf{n}}_k}\}, k = 0, \dots, N_{\mathbf{n}}$, has representatives of all degrees from $m\overline{n}$ up to $|\mathbf{n}| + D$, there exists a unique system of constants $\lambda^*_{\mathbf{n},k}$, $k = 0 \dots, N_{\mathbf{n}}$, such that

$$\deg(TQ_{\mathbf{n}} - \sum_{k=0}^{N_{\mathbf{n}}} \lambda^*_{\mathbf{n},k} L_{\overline{\mathbf{n}}_k}) \leq m\overline{n} - 1 \,. \tag{4.40}$$

From (4.38) and (4.37) it follows that

$$0 = \int x^\nu \left(TQ_{\mathbf{n}} - \sum_{k=0}^{N_{\mathbf{n}}} \lambda^*_{\mathbf{n},k} L_{\overline{\mathbf{n}}_k} \right)(x) ds_{1,j}(x), \quad \nu = 0, \dots, \overline{n} - 1, \quad j = 1, \dots, m.$$

Consequently, using (4.40) we obtain that

$$TQ_{\mathbf{n}} - \sum_{k=0}^{N_{\mathbf{n}}} \lambda^*_{\mathbf{n},k} L_{\overline{\mathbf{n}}_k} \equiv 0 \,,$$

which is (4.39). The last statement follows from the fact that $TQ_{\mathbf{n}}$ is monic. □

In order to make use of the ratio asymptotics of the type II multiorthogonal polynomials of a Nikishin system we will restrict our attention in the following to Nikishin systems $(s_{1,1},\ldots,s_{1,m}) = \mathcal{N}(\sigma_1,\ldots,\sigma_m)$ such that the generating measures $\sigma_j, j = 1,\ldots,m$ have compact support. Additionally, if $\Delta_j = \mathrm{Co}(\mathrm{supp}(\sigma_j))$ we will also assume that $\Delta_j \cap \Delta_{j+1} = \emptyset, j = 1,\ldots,m-1$ and σ_j' a.e on Δ_j. We denote this by writing $(s_{1,1},\ldots,s_{1,m}) = \mathcal{N}'(\sigma_1,\ldots,\sigma_m)$

We are ready to prove convergence of type II Hermite Padé approximants in 1-Hausdorff content.

Theorem 4.3.3. *Let* $\mathbf{s} = (s_{1,1},\ldots,s_{1,m}) = \mathcal{N}'(\sigma_1,\ldots,\sigma_m)$ *and* $\mathbf{r} = (r_1,\ldots,r_m)$ *be given, where all the poles of the rational functions lie in* $\mathbb{C}\setminus(\Delta_1)$. *Let* $\Lambda \subset \mathbb{Z}_+^m$ *be an infinite sequence of distinct multi-indices satisfying condition (3.14). Let* $\left\{\frac{P_{\mathbf{n},1}}{Q_{\mathbf{n}}},\ldots,\frac{P_{\mathbf{n},m}}{Q_{\mathbf{n}}}\right\}$, $\mathbf{n}\in\Lambda$, *be a corresponding sequence of type II Hermite-Padé approximants of* $\mathbf{f} = \mathbf{s}+\mathbf{r}$. *Then for* $j = 1\ldots,m$

$$h - \lim_{\mathbf{n}\in\Lambda} \frac{P_{\mathbf{n},j}}{Q_{\mathbf{n}}} = f_j = \widehat{s}_{1,j} + r_j, \qquad \textit{inside} \qquad (\mathbb{C}\setminus\Delta_1), \tag{4.41}$$

Proof. By Lemma 4.3.2

$$TQ_{\mathbf{n}} = \sum_{k=0}^{N_{\mathbf{n}}} \lambda_{\mathbf{n},k}^* L_{\mathbf{n}_k}.$$

Take

$$\lambda_{\mathbf{n}} = \left(\sum_{k=0}^{N_{\mathbf{n}}} \lambda_{\mathbf{n},k}^*\right)^{-1}, \qquad \lambda_{\mathbf{n},k} = \lambda_{\mathbf{n},k}^*\lambda_{\mathbf{n}}, \quad k = 0,\ldots N_{\mathbf{n}},$$

(since $Q_{\mathbf{n}} \not\equiv 0, \lambda_{\mathbf{n}}$ is finite). Set $H_{\mathbf{n}} := \lambda_{\mathbf{n}} TQ_{\mathbf{n}}$. We have

$$\Psi_{\mathbf{n}} = \frac{H_{\mathbf{n}}}{L_{\mathbf{n}_0}} = \sum_{k=0}^{N_{\mathbf{n}}} \lambda_{\mathbf{n},k} \frac{L_{\overline{\mathbf{n}}_k}}{L_{\overline{\mathbf{n}}_0}}, \qquad \sum_{k=0}^{N_{\mathbf{n}}} |\lambda_{\mathbf{n},k}| = 1. \tag{4.42}$$

From (4.33) it follows that for each $k \geq 0$

$$\lim_{\mathbf{n}\in\Lambda} \frac{L_{\overline{\mathbf{n}}_k}}{L_{\overline{\mathbf{n}}_0}}(z) = \psi^k(z),$$

uniformly on compact subsets of $\mathbb{C}\setminus\Delta_1$

The function ψ is holomorphic and one to one in $\mathbb{C} \setminus \Delta_1$. Consequently, the sequence $\Psi_{\mathbf{n}}$ is uniformly bounded on each compact subset of $\mathbb{C} \setminus \Delta_1$. From the same relations it follows that any limit function of the sequence $\{\Psi_{\mathbf{n}}\}$ is a polynomial of degree at most

$$M := mC + (m+1)D$$

of $\psi(z)$, where C is the constant which appears in (3.14).

As a consequence of these remarks, let us show that on any compact $K \subset \mathbb{C} \setminus \Delta_1$, for all sufficiently large multi-index $|\mathbf{n}|, \mathbf{n} \in \Lambda$, there lie no more than M zeros of the polynomial $TQ_{\mathbf{n}}$. In fact, suppose that this is not so; then, there exists a compact $K \subset \mathbb{C} \setminus \Delta_1$and a sequence $\Lambda' \subset \Lambda$ such that the number of zeros of $TQ_{\mathbf{n}}$ on K for $\mathbf{n} \in \Lambda'$ is greater than M. Let us take a sequence $\Lambda_1 \subset \Lambda'$ such that

$$\lim_{\mathbf{n} \in \Lambda_1} \Psi_{\mathbf{n}}(z) = \Psi_{\Lambda_1}(z) = \sum_{k=0}^{M} \lambda_k \psi^k(z), \quad \sum_{k=0}^{M} |\lambda_k| = 1,$$

uniformly on compact subsets of $\mathbb{C} \setminus \Delta_1$.

Since ψ is one to one on $\mathbb{C} \setminus \Delta_1$, Ψ_{Λ_1} has $\leq M$ zeros on K, and consequently by Rouche's theorem the same is true for the polynomials $TQ_{\mathbf{n}}$ for all $\mathbf{n} \in \Lambda'$, with $|\mathbf{n}|$ sufficiently large. This contradiction implies that the statement made on the number of zeros of $TQ_{\mathbf{n}}$ is true.

Fix a compact $K \subset \mathbb{C} \setminus \Delta_1$. Let $\delta > 0$ be sufficiently small so that the δ-neighborhood K_δ of K is contained in $\mathbb{C} \setminus \Delta_1$ together with its closure.

Let $g_{\mathbf{n}}$ be the monic polynomial whose zeros are the zeros of $TQ_{\mathbf{n}}$ that lie on K_δ. By virtue of what was said above, for all sufficiently large $|\mathbf{n}|, \mathbf{n} \in \Lambda$ we have that $\deg g_{\mathbf{n}} := d'_{\mathbf{n}} \leq M$.

Multiplying (4.35) by $g_{\mathbf{n}}(z)/TQ_{\mathbf{n}}(z)$ and using Lemma 4.3.2, for all $j = 1, \ldots, m$ we obtain

$$g_{\mathbf{n}}(z) \left(f_j(z) - \frac{P_{\mathbf{n},j}(z)}{Q_{\mathbf{n}}(z)} \right) = \frac{g_{\mathbf{n}}(z)}{\lambda_{\mathbf{n}} TQ_{\mathbf{n}}(z)} \int \frac{(\lambda_{\mathbf{n}} TQ_{\mathbf{n}})(x)}{z-x} ds_{1,j}(x), \tag{4.43}$$

$$= g_{\mathbf{n}}(z) \frac{L_{\overline{\mathbf{n}}_0}(z)}{H_{\mathbf{n}}(z)} \sum_{k=0}^{N_{\mathbf{n}}} \lambda_{\mathbf{n},k} \frac{L_{\overline{\mathbf{n}}_k}}{L_{\overline{\mathbf{n}}_0}}(z) I_{\mathbf{n},k}(z), \tag{4.44}$$

where

$$I_{\mathbf{n},k}(z) = \int \frac{L_{\overline{\mathbf{n}}_k}(x)}{L_{\overline{\mathbf{n}}_k}(z)} \frac{ds_{1,j}(x)}{z-x}$$

However, from (4.15) it follows that $I_{\mathbf{n},k}$ is the remainder of the Hermite-Padé approximant of $\hat{s}_{1,j}$ with respect to the multi-index $\mathbf{n}_k$. From [32, Corollary 1.1] we know that

$$\lim_{\mathbf{n}\in\Lambda} I_{\mathbf{n},k}(z) = 0, \quad z \in \mathbb{C} \setminus \Delta_1, \quad k = 1, \ldots (m+1)D, \tag{4.45}$$

From what was said before, it is obvious that the sequence $\left\{g_{\mathbf{n}}(z)\frac{L_{\mathbf{n}_0}(z)}{H_{\mathbf{n}}(z)}\right\}$, $\mathbf{n} \in \Lambda$, of analytic functions on K_δ, is uniformly bounded on K. On account of these remarks, from (4.44) and (4.45) it follows that

$$\lim_{\mathbf{n}\in\Lambda} g_{\mathbf{n}}(z)\left(f_j(z) - \frac{P_{\mathbf{n},j}(z)}{Q_{\mathbf{n}}(z)}\right) = 0, \qquad z \in K. \tag{4.46}$$

Fix $\varepsilon > 0$ and denote $R_{\mathbf{n}} = \left\|\varepsilon^{-1}|g_n(z)|\left|(\frac{P_{\mathbf{n},j}}{Q_{\mathbf{n}}} - f)(z)\right|\right\|_K$. For $z \in K$ we have that $R_{\mathbf{n}} \geq \varepsilon^{-1}|g_n(z)|\left|(\frac{P_{\mathbf{n},j}}{Q_{\mathbf{n}}} - f)(z)\right|$. Then, using the monotonicity of the Hausdorff content it follows that

$$h\left(\left\{z \in K : \left|(\frac{P_{\mathbf{n},j}}{Q_{\mathbf{n}}} - f)(z)\right| \geq \varepsilon\right\}\right) \tag{4.47}$$

$$= h\left(\left\{z \in K : \frac{|g_n(z)|\left|(\frac{P_{\mathbf{n},j}}{Q_{\mathbf{n}}} - f)(z)\right|}{\varepsilon} \geq |g_n(z)|\right\}\right) \tag{4.48}$$

$$\leq h\left(\{z \in K : |g_n(z)| \leq R_{\mathbf{n}}\}\right). \tag{4.49}$$

Since $h\left(\{z : |g_{\mathbf{n}}(z)| \leq R_{\mathbf{n}}\}\right) \leq d'_{\mathbf{n}} R_{\mathbf{n}}^{1/d'_{\mathbf{n}}}$ we arrive at

$$h\left(\left\{z \in K : \left|(\frac{P_{\mathbf{n},j}}{Q_{\mathbf{n}}} - f)(z)\right| \geq \varepsilon\right\}\right) \leq d'_{\mathbf{n}} R_{\mathbf{n}}^{1/d'_{\mathbf{n}}}. \tag{4.50}$$

Let $\Lambda' \subset \Lambda$ be such that $\deg g_{\mathbf{n}} = d'_{\mathbf{n}} = 0$ for $\mathbf{n} \in \Lambda'$. If Λ' contains infnitely many multi-indices from (4.46) we obtain directly that

$$\lim_{\mathbf{n}\in\Lambda'} \left(f_j(z) - \frac{P_{\mathbf{n},j}(z)}{Q_{\mathbf{n}}(z)}\right) = 0, \quad z \in K. \tag{4.51}$$

For the rest of the multi-indices in Λ we have that $1 \leq d'_{\mathbf{n}} \leq M$. Combining this with (4.46) and (4.50) we obtain that

$$\lim_{\mathbf{n}\in\Lambda} h\left(\left\{z \in K : \left|f_j(z) - \frac{P_{\mathbf{n},j}(z)}{Q_{\mathbf{n}}(z)} \geq \varepsilon\right|\right\}\right) = 0$$

which is what we needed to prove. □

In order to derive uniform convergence on compact subsets of $(\mathbb{C} \setminus \Delta_1)'$ using Gonchar's lemma, we would need to have a better control on the number of zeros that $Q_{\mathbf{n}}$ may have wandering around in $\mathbb{C} \setminus \Delta_1$. More precisely, we would need to know, for example, that for any $\varepsilon > 0$ sufficiently small there exists an N such that for all $\mathbf{n} \in \Lambda$, $|\mathbf{n}| > N$, the number of zeros outside the ε-neighborhood of Δ_1 is at most $D = \deg T$. However, we only know that it is at most $m(C + D)$ (D zeros of $TQ_{\mathbf{n}}$ are due to T). Even if $C = 0$ in (3.14), we only have the correct bound for the scalar case $m = 1$. But this case is well known. For the time being this problem remains open. Comparing with the results we have for the case when the r_j have real coefficients we conjecture that the correct bound can be obtained at least when the poles of the rational functions $r_j, j = 1, \ldots m$ are different for distinct j.

Chapter 5

Conclusions and further work

In this dissertation we have provided a wide class of perfect function matrices and obtained several results on the location and interlacing properties of the zeros of the mixed type Hermite-Padé approximants. In particular, for type I Hermite-Padé approximants with respect to a Nikishin system and for type II Hermite-Padé aproximants with respect to a perturbed Nikishin system we obtained the location of their zeros.

Possibly, the most significant result of this work is Theorem 3.2.1. For the first time a Markov-Stieltjes type theorem is given for type I Hermite-Padé approximation. It clarifies in this case what should be understood as the approximants and what are the functions being approximated. Moreover, Theorem 4.1.2 is a nice consequence of it. In turn, Theorem 4.1.2 plays a significant role in the proof of Theorem 4.2.5 which is another important result of the thesis.

For the future there are some questions we would like to address.

- Using type I and type II Hermite-Padé approximation one can recover the generating measures σ_1 and σ_m of a Nikishin system. What type of construction allows to recover directly, through rational approximation the rest of the generating measures?

- Study the interlacing properties of the polynomials $a_{\mathbf{n},k}$ for " consecutive" multi-indices.

- In Theorem 4.1.1 and Theorem 4.2.5 an important condition is that the polynomial perturbations (rational pertubation) have no common zeros (poles), respectively. What happens if this condition is omitted? Do these theorems remain valid? Is there any counterexample?

- When the rational perturbations have complex coefficients, find sufficient conditions in order to obtain in Theorem 4.3.3 uniform convergence on compact subsets instead of convergence in Hausdorff content.
- Study the ratio asymptotics of general sequences of " consecutive" type II multiple orthogonal polynomials (see Lemma 4.3.1). A positive result would allow to improve the bound on the number of zeros of $Q_{\mathbf{n}}$ away from Δ_1, in the proof of Theorem 4.3.3.
- Study the convergence of type I and type II Fourier Hermite-Padé approximation of Nikishin system in which the rational approximants are constructed from orthogonality conditions instead of interpolation.

Bibliography

[1] L. V. Ahlfors, *Complex Analysis*, third ed., International Series in Pure and Applied Mathematics. McGraw-Hill Book Co., New York, 1978.

[2] C. Álvarez-Fernández, U. Fidalgo, and M. Mañas, *The multicomponent 2D Toda hierarchy: generalized matrix orthogonal polynomials, multiple orthogonal polynomials and Riemann-Hilbert problems*, Inverse Problems **26** (2010), no. 5, 055009, 15.

[3] C. Álvarez-Fernández, U. Fidalgo, and M. Mañas, *Multiple orthogonal polynomials of mixed type: Gauss-Borel factorization and the multi-component 2D Toda hierarchy*, Advances in Mathematics **227** (2011), no. 4, 1451–1525.

[4] A. Angelesco, *Sur deux extensions des fractions continues algébriques*, C.R. Acad. Sci. Paris **18** (1919), 262–263.

[5] R. Apery, *Irrationalité de $\zeta(2)$ et $\zeta(3)$*, Astérisque **61**(1979), 11–13.

[6] A. I. Aptekarev, *Asymptotics of simultaneously orthogonal polynomials in the Angelesco case*, Math. USSR. Sb. **64** (1989), no. 1, 57–84.

[7] A. I. Aptekarev, *Sharp constants for rational approximations of analytic functions*, Sbornik: Mathematics . **193** (2002), no. 1, 1–72.

[8] A. I. Aptekarev, G. López Lagomasino, and I. A. Rocha, *Ratio asymptotics of Hermite-Padé polynomials for Nikishin systems*, Matematicheskii Sbornik, **196** (2005), 1089–1107.

[9] G. A. Baker, *The theory and application of the Padé approximant method*, Advances in Theoretical Physics, Vol. 1, Academic Press, New York, 1965, pp. 1–58.

[10] G. A. Baker, *The Padé Approximant in Theoretical Physics*, Academic Press, New York-London, 1970, Mathematics in Science and Engineering, Vol. 71.

[11] M. Bertola, M. Gekhtman, and J. Szmigielski, *Cauchy biorthogonal polynomials*, J. Approx. Theory **162** (2010), no. 4, 832–867.

[12] F. Beukers, *A note on the irrationality of $\zeta(2)$ and $\zeta(3)$*, Bull. London Math. Soc. **11** (1979), no. 3, 268–272.

[13] A. Borodin, *Biorthogonal ensembles*, Nuclear Phys. B **536** (1999), no. 3, 704–732.

[14] P. M. Bleher and A. B. J. Kuijlaars, *Random matrices with external source and multiple orthogonal polynomials*, Int. Math. Res. Not. (2004), no. 3, 109–129.

[15] J. Bustamante and G. López Lagomasino, *Hermite-Padé approximations for Nikishin systems of analytic functions*, Russian Acad. Sci. Sb. Math. **77** (1994), no. 2, 367–384.

[16] J. Bustamante and G. López Lagomasino, *Hermite-Padé approximation to a Nikishin type system of meromorphic functions*, J. Approx. Theory **81** (1995), no. 1, 1–12.

[17] T. Carleman, *Les fonctions quasi-analytiques.*, Gauthiers Villars, Paris 1926.

[18] J. Coates, *On the algebraic approximation of functions.*, Nederl. Akad. Wetensch. Proc. Ser. A 69=Indag. Math. **28** (1966), 449–461.

[19] J. Coussement and W. Van Assche, *Gaussian quadrature for multiple orthogonal polynomials*, J. Comput. Appl. Math. **178** (2005), no. 1-2, 131–145.

[20] E. Daems and A. B. J. Kuijlaars, *A Christoffel-Darboux formula for multiple orthogonal polynomials*, J. Approx. Theory **130** (2004), no. 2, 190–202.

[21] E. Daems and A. B. J. Kuijlaars, *Multiple orthogonal polynomials of mixed type and non-intersecting Brownian motions*, J. Approx. Theory **146** (2007), no. 1, 91–114.

[22] E. Daems, A. B. J. Kuijlaars, and W. Veys, *Asymptotics of non-intersecting Brownian motions and a* 4×4 *Riemann-Hilbert problem*, J. Approx. Theory **153** (2008), no. 2, 225–256.

[23] S. Delvaux, *Average characteristic polynomials for multiple orthogonal polynomial ensembles*, J. Approx. Theory **162** (2010), no. 5, 1033–1067.

[24] S. Delvaux and A. B. J. Kuijlaars, *A graph-based equilibrium problem for the limiting distribution of nonintersecting Brownian motions at low temperature*, Constr. Approx. **32** (2010), no. 3, 467–512.

[25] P. Desrosiers and P. J. Forrester, *A note on biorthogonal ensembles*, J. Approx. Theory **152** (2008), no. 2, 167–187.

[26] K. Driver and H. Stahl, *Normality in Nikishin systems*, Indag. Math. (N.S.) **5** (1994), no. 2, 161–187.

[27] K. Driver and H. Stahl, *Simultaneous rational approximants to Nikishin-systems. I*, Acta Sci. Math. (Szeged) **60** (1995), no. 1-2, 245–263.

[28] K. Driver and H. Stahl, *Simultaneous rational approximants to Nikishin systems. II*, Acta Sci. Math. (Szeged) **61** (1995), no. 1-4, 261–284.

[29] U. Fidalgo, A. López García, G. López Lagomasino, and V. N. Sorokin, *Mixed type multiple orthogonal polynomials for two Nikishin systems*, Constr. Approx. **32** (2010), no. 2, 255–306.

[30] U. Fidalgo and G. López Lagomasino, *Rate of convergence of generalized Hermite-Padé approximants of Nikishin systems*, Constr. Approx. **23** (2006), no. 2, 165–196.

[31] U. Fidalgo and G. López Lagomasino, *General results on the convergence of multipoint Hermite-Padé approximants of Nikishin systems*, Constr. Approx. **25** (2007), no. 1, 89–107.

[32] U. Fidalgo and G. López Lagomasino, *Nikishin systems are perfect*, Constr. Approx. **34** (2011), no. 3, 297–356.

[33] U. Fidalgo and G. López Lagomasino, *Nikishin systems are perfect. The case of unbounded and touching supports*, J. Approx. Theory **163** (2011), no. 6, 779–811.

[34] U. Fidalgo, G. López Lagomasino, and S Medina Peralta, *Hermite-Padé approximation for certain systems of meromorphic functions*, ArXiv: 1310.7010v1(Submitted) (2013).

[35] U. Fidalgo, S. Medina Peralta, and J. Mínguez Ceniceros, *Mixed type multiple orthogonal polynomials: perfectness and interlacing properties of zeros*, Linear Algebra Appl. **438** (2013), no. 3, 1229–1239.

[36] A. A. Gonchar, *On the convergence of Padé approximants for some class of meromorphic functions*, Math. USSR Sb. **26** (1975), no. 4, 555–575.

[37] A. A. Gonchar, *On the convergence of generalized Padé approximants of meromorphic functions*, Math. USSR Sb. **27** (1975), no.4, 503–514.

[38] A. A. Gonchar, *Rational approximations of analytic functions*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986), Amer. Math. Soc., Providence, RI, 1987, 739–748.

[39] A. A. Gonchar and G. López Lagomasino, *On Markov's theorem for multipoint Padé approximants*, Math. USSR Sb. **34** (1978), no. 4, 449–459.

[40] A. A. Gonchar and E. A. Rakhmanov, *On the convergence of simultaneous Padé approximants for systems of functions of Markov type*, Proceedings of the Steklov Institute of Mathematics. no. 3 (1983), 31–50.

[41] A. A. Gonchar and E. A. Rakhmanov, *Equilibrium distributions and degree of rational approximation of analytic functions*, Math. USSR. Sb. **62** (1989), no. 2, 305–352.

[42] A. A. Gonchar, E. A. Rakhmanov, and V. N. Sorokin, *Hermite-Padé approximants for systems of functions of Markov type*, Sbornik: Mathematics **188** (1997), no. 5, 671–696.

[43] M. Haneczok and W. Van Assche, *Interlacing properties of zeros of multiple orthogonal polynomials*, J. Math. Anal. Appl. **389** (2012), no. 1, 429–438.

[44] Ch Hermite, *Sur la fonction exponentielle*, C. R. Acad. Sci. Paris **77** (1873), 18–24, 74–79, 226–233, 285–293.

[45] H. Jager, *A multidimensional generalization of the Padé table.*, Nederl. Akad. Wetensch. Proc. Ser. A 67=Indag. Math. **26** (1964), 227–239, 240–244, 245–249.

[46] M. G. Krein and A. A. Nudel´man, *The Markov Moment Problem and Extremal Problems*, American Mathematical Society,Translations of Mathematical Monographs, Vol. 50. Providence. R.I, 1977.

[47] A. B. J. Kuijlaars, *Multiple orthogonal polynomial ensembles*, Recent trends in orthogonal polynomials and approximation theory, Contemp. Math., Vol. 507, Amer. Math. Soc., Providence, RI, 2010, 155–176.

[48] A. B. J. Kuijlaars, *Multiple orthogonal polynomials in random matrix theory*, Proceedings of the International Congress of Mathematicians. Volume III, Hindustan Book Agency, New Delhi, 2010, 1417–1432.

[49] A. López García and G. López Lagomasino, *Ratio asymptotic of Hermite-Padé orthogonal polynomials for Nikishin systems. II*, Adv. Math. **218** (2008), no. 4, 1081–1106.

[50] G. López Lagomasino, *On the convergence of multipoint Padé approximants for functions of Stieltjes type.*, Soviet Math. Dokl. **19** (1978), 425–428.

[51] G. López Lagomasino, *The convergence of Padé approximants for meromorphic functions of Stieltjes type*, Math. USSR. Sb. **39** (1981), no. 2, 281–288.

[52] G. López Lagomasino, *On the moment problem and the convergence of Padé approximants for meromorphic functions of Stieltjes type*, Constructive function theory '81 (Varna, 1981), Bulgar. Acad. Sci., Sofia, 1983, pp. 419–424.

[53] G. López Lagomasino, *Convergence of Padé approximants for meromorphic functions of Stieltjes type and comparative asymptotics for orthogonal polynomials*, Math.USSR Sb. **64** (1989), no. 1, 207–229.

[54] G. López Lagomasino and A. Martínez Finkelshtein, *Rate of convergence of two-point Padé approximants and logarithmic asymptotics of Laurent-type orthogonal polynomials*, Constr. Approx. **11** (1995), no. 2, 255–286.

[55] G López Lagomasino and S Medina Peralta, *On the convergence of type I Hermite-Padé approximants*, ArXiv: 1307.0213v1(Submitted) (2013).

[56] K. Mahler, *Perfect systems*, Compositio Math. **19** (1968), 95–166 (1968).

[57] A. Markov, *Deux démonstrations de la convergence de certaines fractions continues*, Acta Math. **19** (1895), no. 1, 93–104.

[58] E. M. Nikishin, *A system of Markov functions*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. (1979), no. 4, 60–63, 103.

[59] E. M. Nikishin, *Simultaneous Padé approximants*, Math. USSR. Sb. **41** (1982), no. 4, 409–425.

[60] E. M. Nikishin, *Asymptotic of linear forms for simultaneous Padé approximants*, Soviet Math (Izv. VUZ) **30** (1986), 43–52.

[61] E. M. Nikishin and V. N. Sorokin, *Rational Approximations and Orthogonality*, Translations of Mathematical Monographs, vol. 92, American Mathematical Society, Providence, RI, 1991.

[62] H. Padé, *Sur la représentation approchée d'une fonction par des fractions rationnelles*, Ann. Sci. École Norm. Sup. **9** (1892), 3–93.

[63] E. A. Rakhmanov and S. P. Suetin, *Asymptotic behavior of Hermite-Padé polynomials of the first kind for a pair of functions forming a Nikishin system*, Uspekhi Mat. Nauk **67** (2012), no. 5, 177–178.

[64] E. A. Rakhmanov and S. P. Suetin, *Distribution of zeros of Hermite-Padé polynomials for a pair of functions forming a Nikishin system*, Sb. Math. (submitted).

[65] E. A Rakhmanov, *Convergence of diagonal Padé approximants*, Math. USSR. Sb. **33** (1977), no. 2, 243–260.

[66] E. A. Rakhmanov, *On the asymptotic of the ratio of orthogonal polynomials*, Math. USSR. Sb. **32** (1977), 199-213.

[67] E. A. Rakhmanov, *On the asymptotic of the ratio of orthogonal polynomials II*, Math. USSR. Sb. **46** (1983), 105-117.

[68] E. A. Rakhmanov, *On asymptotic properties of orthogonal polynomials on the unit circle with weights not satisfying Szegő's condition*, Math. USSR. Sb. **58** (1987), 149-167.

[69] T. Ransford, *Potential theory in the complex plane*, London Mathematical Society Student Texts, vol. 28, Cambridge University Press, Cambridge, 1995.

[70] W. Rudin, *Real and complex analysis*, third ed., McGraw-Hill Book Co., New York, 1987.

[71] V. N. Sorokin, *Simultaneous approximation of several linear forms*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. (1983), no. 1, 44–47.

[72] H. Stahl, *Simultaneous rational approximants*, **5** (1995), 325–349.

[73] H. Stahl, *Best uniform rational approximation of x^α on* $[0,1]$, Acta Math. **190** (2003), no. 2, 241–306.

[74] H. Stahl and V. Totik, *General Orthogonal Polynomials*, Encyclopedia of Mathematics and its Applications, vol. 43, Cambridge University Press, Cambridge, 1992.

[75] T. J. Stieltjes, *Recherches sur les fractions continues*, Ann. Fac. Sci. Toulouse Sci. Math. Sci. Phys. **8** (1894), no. 4, J1–J122.

[76] T. J. Stieltjes, *Recherches sur les fractions continues [Suite et fin]*, Ann. Fac. Sci. Toulouse Sci. Math. Sci. Phys. **9** (1895), no. 1, A5–A47.

[77] P. Tchebycheff, *Sur les fractions continues.*, Usp. Zap. Imper. Akad. Nauk. **3** (1855), 636–664.

[78] P. Tchebycheff, *Sur les values limites des integrales*, J. Math. Pures Appl. **19** (1874), 2052–2053.

[79] P. Tchebycheff, *Sur les developpement des fonctions a une seule variable.*, Bull. Phys-Math. Acad. Imp. Sci. St. Petersburg **1** (1895), 193–200.

[80] W. Van Assche, *Analytic number theory and rational approximation*, Coimbra Lecture Notes on Orthogonal Polynomials. Nova Science Pub, New York (2008), 197–229.